Nahm transform for integrable connections on the Riemann sphere

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CHAPTER 1

Introduction

1.1. Historical context and abstract of the thesis

Nahm transform is a non-linear analog for instantons of the usual Fourier transform on functions. It has been extensively studied starting from the beginning of the 1980's, inspired by the seminal work of M. F. Atiyah, V. Drinfeld, N. J. Hitchin and Yu. I. Manin on a correspondence (the *ADHM-transform*) between finite-energy solutions of the *Yang-Mills equations* and some algebraic data (see [**1**], [**10**]). The Yang-Mills equations are the anti-self-duality equations for a unitary connection on a Hermitian vector bundle defined over $\mathbf{R}^4$; their finite-energy solutions are called *instantons*.

Since then, it turned out that the general picture concerning this correspondence is as follows: let X be any manifold obtained as a quotient of $\mathbf{R}^4$ by a closed additive subgroup Λ. The solutions of the Yang-Mills equations invariant by Λ (that are clearly not of finite energy in the case $\Lambda \neq \{0\}$) can be identified in an obvious manner to solutions of a system of differential equations on X, called the *reduction* of the Yang-Mills equations. On the other hand, denoting by $(\mathbf{R}^4)^*$ the dual of the vector space $\mathbf{R}^4$, Λ determines a closed additive subgroup Λ^* called the *dual subgroup* by saying that an element $\xi \in (\mathbf{R}^4)^*$ is in Λ^* if and only if $\xi(\lambda) \in \mathbf{Z}$ for all $\lambda \in \Lambda$. Hence, we can form the *dual manifold* $X^* = (\mathbf{R}^4)^*/\Lambda^*$ of X, that also admits a reduction of the Yang-Mills equations. Nahm transform is then a procedure that maps solutions of the reduced equations on X to solutions of the reduced equations on X^* bijectively up to overall gauge transformations on both sides. One remarks that there is a canonical isomorphism between $((\mathbf{R}^4)^*)^*$ and $\mathbf{R}^4$, as well as between $(\Lambda^*)^*$ and Λ. Therefore, if we start from a solution of the reduced equations on X and iterate Nahm transform twice, we again get a solution of the reduced equations on X. One important property analogous to usual Fourier transform is that in some cases the solution we get this way is, up to a coordinate change $x \mapsto -x$,

known to be the solution we started with; that is, Nahm transform is (up to a sign) *involutive.* Moreover, in some cases one knows that the moduli spaces of solutions of the reduced equations modulo gauge transformations on X and on X^* are smooth hyper-Kähler manifolds with respect to the metric induced by L^2-norm and the complex structures induced by $\mathbf{R}^4$; Nahm transform is then a hyper-Kähler isometry between these moduli spaces. This is to be compared with Parseval's theorem which states that usual Fourier transform defines an isometry between L^2-spaces of functions.

Putting $\Lambda = \{0\}$, one gets $X = \mathbf{R}^4$ and $\Lambda^* = \mathbf{R}^4$, so $X^* = \{0\}$. In this case, Nahm transform reduces to the ADHM-transform. There are several other examples of Nahm transform in the literature for different subgroups of $\mathbf{R}^4$; for a nice exposition of these, see the survey paper [14] of M. Jardim. In this work, we are concerned with the case $\Lambda = \mathbf{R}^2$. In this case, the base manifold is $X = \mathbf{R}^2$, and its dual X^* is another copy of the real plane that we shall denote by $\hat{\mathbf{R}}^2$. These are non-compact manifolds, with compactifications the Riemann spheres $\mathbf{CP}^1$ and $\widehat{\mathbf{CP}}^1$ respectively. The reduction of the original (Yang-Mills) equations can be viewed in two different ways depending on the complex structure that we choose: they are the equations defining an integrable connection with harmonic metric, or equivalently, those defining a Higgs bundle with Hermitian-Einstein metric. Now, it turns out that there are no smooth solutions on the Riemann sphere of either one of these equations except for the trivial ones (c.f. [12]). However, there are solutions having prescribed singularities in some points, and the solutions of one equation are still in correspondence with those of the other: this is proved by O. Biquard and Ph. Boalch in [6]. We establish, under some hypotheses on the singularity behavior, Nahm transform for singular integrable connections (or equivalently, singular Higgs bundles) on the Riemann sphere. Note that Nahm transform for singular objects have already been studied by O. Biquard and M. Jardim in [7] and by S. A. Cherkis and A. Kapustin in [8]. On the other hand, using different techniques, B. Malgrange has defined in [18] a so-called Fourier-Laplace transform for integrable connections with singularities on the Riemann sphere behaving in the same manner on the level of singularity data as the one we define here. It is therefore very natural to believe that these two transforms actually agree. One difference between these works is,

however, the transformation of a parabolic structure and an adapted harmonic metric at the singularities in our case; for details, see Section 1.4.

The construction follows the main ideas of other Nahm transforms found in literature. Namely, in Section 2.1 we define positive and negative spinor bundles $S^\pm$ over $\mathbf{CP}^1$, as well as a Dirac operator

$$\slashed{\partial} : S^+ \otimes E \longrightarrow S^- \otimes E.$$

We then let $\xi \in \hat{\mathbf{C}} \smallsetminus \hat{P}$ be a parameter, where $\hat{P}$ is the singular locus of the transformed objects, and for all ξ twist the operator $\slashed{\partial}$ by some flat connection to obtain a family of operators $\slashed{\partial}_\xi$. In Section 2.2 we prove that the kernel of these twisted operators vanish and that the cokernels form a finite-dimensional space. Furthermore, this dimension is independent of ξ; we then define the transformed vector bundle $\hat{E}$ on $\hat{\mathbf{C}}$ as the vector bundle with fiber over ξ given by $coKer(\slashed{\partial}_\xi)$. In Section 2.3 we carry out an analog of L^2-Hodge theory of a compact Kählerian manifold in this case; namely we establish an isomorphism between this cokernel and the first L^2-cohomology of an elliptic complex, as well as harmonic 1-forms with respect to the Laplacian of the Dirac operator. We then go on to define the transformed flat bundle and the transformed Hermitian metric in Section 3.1, and we extend the flat bundle over the singularities – so defining the transformed meromorphic integrable connection – in Section 3.2. The transformed metric is then shown to be Hermitian-Einstein in Section 4.2. Next, in Section 4.3 we give a completely explicit description of the fibers of the transformed bundle, first in terms of hypercohomology of a sheaf map, then in terms of the corresponding spectral set. Then come the constructions of the extensions of the transformed Higgs bundle to the singular points (Section 4.4). This allows us to obtain the singularity data of the transformed Higgs bundle in Sections 4.5 and 4.6, and we complete the transform by computing the topology of the transformed Higgs bundle in Section 4.7. Finally, Chapter 5 deals with the involutivity property of the transform.

1.2. Integrable connection point of view

Let $\mathbf{C}$ be the complex line, with its natural holomorphic coordinate $z = x + iy$ and Euclidean metric $|\mathrm{d}z|^2$; and let $\mathbf{CP}^1$ be the complex projective line. Let $E \to \mathbf{CP}^1$ be a rank r holomorphic vector bundle

on the Riemann sphere, and D be a meromorphic integrable connection on it, with *first order* or *logarithmic singularities* at the points of a finite set $\{p_1, \ldots, p_n\} = P \subset \mathbf{C}$ and a second order singularity at infinity. In other words, on a small disk $\Delta(p_j, \varepsilon)$ centered at $p_j \in P$ in a holomorphic basis $\{\tau_k^j\}_{k=1,\ldots,r}$ of E, D is of the form $D^j + b^j$ where b^j is a holomorphic 1-form on the disk and

$$(1.1) \qquad\qquad D^j = \mathrm{d} + \frac{A^j}{z - p_j} \mathrm{d}z \wedge .$$

We suppose furthermore that A^j is diagonal:

$$A^j = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \mu_{r_j+1}^j & & \\ & & & & \ddots & \\ & & & & & \mu_r^j \end{pmatrix};$$

it is called the *residue* of D at p_j, and $1 \le r - r_j \le r$ is the rank of A^j. For convenience, we put $\mu_1^j = \ldots = \mu_{r_j}^j = 0$, so that $A^j = diag(\mu_k^j)_{k=1,\ldots r}$. We will often make use of the holomorphic local decomposition

$$(1.2) \qquad\qquad E^j = E_{reg}^j \oplus E_{sing}^j,$$

into the *regular* and *singular components* of E near p_j; here by definition E_{reg}^j is the holomorphic subbundle of $E^j = E|_{\Delta(p_j,\varepsilon)}$ spanned by $\{\tau_k^j\}_{k=1,\ldots,r_j}$, and E_{sing}^j is the one spanned by $\{\tau_k^j\}_{k=r_j+1,\ldots,r}$. Intrinsically, E_{sing}^j is the sum of the generalized eigenspaces corresponding to all eigenvalues converging to infinity of the integrable connection, whereas E_{reg}^j is the sum of the generalized eigenspaces corresponding to the eigenvalues that remain bounded.

In a similar manner, at infinity D is supposed to be equal (up to a holomorphic term) to a meromorphic local model having a *second order pole*, so that in a holomorphic basis $\{\tau_k^\infty\}_{k=1,\ldots,r}$ on a disk $\mathbf{C} \setminus \Delta(0, R)$ corresponding to a standard neighborhood of infinity in $\mathbf{CP}^1$, it is of the form $D = D^\infty + b^\infty$ where b^∞ is now a holomorphic 1-form in the given neighborhood of infinity, and

$$(1.3) \qquad\qquad D^\infty = \mathrm{d} + \left(A + \frac{C}{z} \right) \mathrm{d}z \wedge$$

is the second order model with diagonal leading term

$$A = \begin{pmatrix} \xi_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \xi_1 & & & & & \\ & & & \ddots & & & & \\ & & & & \ddots & & & \\ & & & & & \xi_{n'} & & \\ & & & & & & \ddots & \\ & & & & & & & \xi_{n'} \end{pmatrix}$$

and residue

$$C = \begin{pmatrix} \mu_1^\infty & & \\ & \ddots & \\ & & \mu_r^\infty \end{pmatrix}.$$

Here $\{\xi_l\}_{l=1}^{n'}$ are the distinct eigenvalues of A. Each ξ_l appears in neighboring positions $k = 1 + a_l, \ldots, a_{l+1}$, in particular its multiplicity is $m_l = a_{l+1} - a_l$. Of course, we must then have $a_1 = 0$ and $a_{n'+1} = r$. In line with the above notation, we set $r_\infty = 0$ and $C = diag(\mu_k^\infty)_{k=1,\ldots,r}$. Furthermore, we will write

$$A = diag(\{\xi_l, m_l\})_{l=1,\ldots,n'}$$

for the diagonal matrix A as given above, meaning that A is diagonal with m_l neighboring eigenvalues equal to ξ_l.

DEFINITION 1.1. *The integrable connections having singularities near the points of $P \cup \{\infty\}$ as described above will be called* meromorphic integrable connections with logarithmic singularities in P and a second-order singularity at infinity, *or for simplicity* meromorphic integrable connections *although they are by far not all the meromorphic integrable connections.*

1.3. The transform of the meromorphic integrable connection

Let (E, D) be a stable vector bundle with a meromorphic integrable connection on the sphere. Our aim in this paper is to define another complex bundle $\hat{E}$ with a meromorphic connection $\hat{D}$ on the sphere out of (E, D), which we call the *transformed meromorphic integrable connection*. Just as the initial connection, the transformed one will

also admit a finite number of simple poles in points of the line and a second-order pole at infinity.

In order to define the transformed vector bundle $\hat{E}$, first we need to set some notation. Let $\hat{\mathbf{C}}$ be another copy of $\mathbf{C}$. (The importance of distinguishing the two copies of $\mathbf{C}$ is to help us avoid confusions.) For a parameter $\xi \in \hat{\mathbf{C}}$, consider the following deformation of D:

$$(1.4) \qquad D_\xi^{int} = D - \xi \mathrm{d}z\wedge,$$

where $\xi : E \to E$ stands for multiplication by ξ. Since we only change the $(1,0)$-part of D, and by an endomorphism that is independent of z, this is then another meromorphic integrable connection, with the same underlying holomorphic bundle as for D. Furthermore, its unitary and self-adjoint parts are given by

$$(1.5) \qquad D_\xi^+ = D^+ - \frac{\xi}{2}\mathrm{d}z + \frac{\bar{\xi}}{2}\mathrm{d}\bar{z}$$

$$(1.6) \qquad \Phi_\xi^{int} = \Phi - \frac{\xi}{2}\mathrm{d}z - \frac{\bar{\xi}}{2}\mathrm{d}\bar{z}.$$

Consider the following family in ξ of elliptic complexes $\mathcal{C}_\xi^{int}$ over $\mathbf{C} \smallsetminus P$:

$$(1.7) \qquad \Omega^0 \otimes E \xrightarrow{D_\xi^{int}} \Omega^1 \otimes E \xrightarrow{D_\xi^{int}} \Omega^2 \otimes E.$$

Fix a Hermitian metric h on E for which the holomorphic sections of the extension at the singularities are bounded (above and below) by a positive constant, and denote by $\hat{E}_\xi^{int}$ the first L^2-cohomology of the complex (1.7) for this metric. In Theorems 2.6 and 2.21 we show that there exists a finite set $\hat{P} \subset \hat{\mathbf{C}}$ such that for $\xi \in \hat{\mathbf{C}} \smallsetminus \hat{P}$ the first L^2-cohomologies of this complex are finite-dimensional of the same dimension for all ξ.

DEFINITION 1.2. *The* transformed vector bundle $\hat{E}$ *is then the vector bundle over* $\hat{\mathbf{C}} \smallsetminus \hat{P}$ *whose fiber over* $\xi \in \hat{\mathbf{C}} \smallsetminus \hat{P}$ *is the first L^2-cohomology* $L^2 H^1(D_\xi^{int})$ *of* $\mathcal{C}_\xi^{int}$.

Let $\xi_0 \in \hat{\mathbf{C}} \smallsetminus \hat{P}$, and let $f(z) \in \hat{E}_{\xi_0}$ be a class in the first cohomology of $\mathcal{C}_{\xi_0}^{int}$.

DEFINITION 1.3. *The* transformed flat connection $\hat{D}$ *is by definition the flat connection whose parallel section* $f(\xi; z)$ *extending* f *in some neighborhood of* ξ_0 *is given by the first L^2-cohomology classes in* $\mathcal{C}_\xi^{int}$ *of*

$$e^{(\xi - \xi_0)z} f(z).$$

Finally, h induces a natural Hermitian metric $\hat{h}$ on $\hat{E}$ as follows: in Theorem 2.21 we show that any class in $L^2 H^1(D_\xi)$ can be represented by a unique harmonic 1-form with respect to the Laplacian of the Dirac operator.

DEFINITION 1.4. *The* transformed Hermitian metric $\hat{h}$ on $\hat{E}$ is defined by the L^2-norm of harmonic representatives.

All this will be explained in more detail in Section 3.1 and in Definition 3.1.

When one considers an integrable connection, there exists sometimes a privileged fiber metric on the bundle, namely a harmonic one. In order to be able to define harmonicity, decompose as usual D into its unitary and self-adjoint part

$$(1.8) \qquad\qquad D = D^+ + \Phi,$$

put ∇_{D^+} or simply ∇^+ for the covariant derivative associated to the connection D^+ (so that $\nabla^+ t$ makes sense for a tensor t of arbitrary type $(T\mathbf{CP}^1)^p \otimes (T^*\mathbf{CP}^1)^q \otimes E^r \otimes (E^*)^s$) and denote by $(\nabla^+)^*_h$ the adjoint operator of ∇^+ with respect to h.

DEFINITION 1.5. *The Hermitian metric h is called* harmonic, *if it satisfies the equation*

$$(1.9) \qquad\qquad (\nabla^+)^*_h \Phi = 0.$$

This is a second-order non-linear partial differential equation in h.

Here is the main result of this thesis in a special case (the one without parabolic structures, see Definition 1.8).

THEOREM 1.6. *Let (E, D, h) be any meromorphic integrable connection with logarithmic singularities in P as in (1.1), and a double pole (1.3) at infinity, endowed with a harmonic metric h. Suppose that the eigenvalues of the polar part of D in the punctures satisfy the following assumptions:*

(1) *for fixed $j \in \{1, \ldots, n\}$, the complex numbers μ^j_k for $k = r_j + 1, \ldots, r$ are all different, and $\Re\mu^j_k \notin \mathbf{Z}$*

(2) *for fixed $l \in \{1, \ldots, n'\}$, the complex numbers μ^∞_k for $k = 1 + a_l, \ldots, a_{l+1}$ are all different, and $\Re\mu^\infty_k \notin \mathbf{Z}$*

Then the set of punctures $\hat{P} \in \hat{\mathbf{C}}$ of the transformed bundle is the set $\{\xi_1, \ldots, \xi_{n'}\}$ of distinct eigenvalues of the leading order term A of D

at infinity. For $\xi \notin \hat{P}$, the first L^2-cohomologies of (1.7) are finite dimensional vector spaces of the same dimension. They match up to define a smooth vector bundle $\hat{E}$ of rank

$$(1.10) \qquad\qquad \hat{r} = \sum_{j=1}^{n}(r - r_j)$$

over $\hat{\mathbf{C}} \smallsetminus \hat{P}$. $\hat{D}$ is a flat connection on $\hat{E}$. It underlies a meromorphic integrable connection (that we continue to denote $(\hat{E}, \hat{D})$) of degree $deg(\hat{E}) = deg(E)$, called the transformed meromorphic connection. *It has logarithmic singularities in $\hat{P}$ and a double pole at infinity. The non-vanishing eigenvalues of the residue in $\xi_l \in \hat{P}$ are $\{-\mu_{1+a_l}^{\infty}, \ldots, -\mu_{a_{l+1}}^{\infty}\}$. The eigenvalues of the second-order term of the transformed meromorphic connection are $\{-p_1, \ldots, -p_n\}$, the multiplicity of $-p_j$ being $(r - r_j)$; the eigenvalues of its residue at infinity on the eigenspace of the second-order term corresponding to $-p_j$ are $\{-\mu_{r_j+1}^{j}, \ldots, -\mu_r^{j}\}$. Finally, $\hat{h}$ is harmonic for $\hat{D}$.*

REMARK 1.7. *The assumptions (1) and (2) of the theorem are clearly generic in the parameter space of all possible eigenvalues.*

This theorem actually follows from the more general statement 1.17. In order to understand the more general setup, one needs to consider meromorphic connections endowed with a parabolic structure.

1.4. Parabolic structure and adapted harmonic metric

Actually, we can suppose more structure on the integrable connection: namely, that it comes with a parabolic structure on P and at infinity.

DEFINITION 1.8. *A* parabolic structure *on (E, D) is the data of a strictly decreasing filtration by vector subspaces*

$$E_p = F_0 E_p \supset F_1 E_p \supset \ldots \supset F_{b_p - 1} E_p \supset F_{b_p} E_p = \{0\}$$

(where $1 \leq b_p \leq r$) of the fiber E_p of E in each singular point $p \in P \cup \{\infty\}$, called the parabolic flag, *such that each F_m is spanned by some of the restrictions $\{\tau_k^{j}(p)\}_{k=1}^{r}$ of the holomorphic basis to the singularity $p = p_j$ or ∞, together with a sequence of corresponding real numbers*

$$0 \leq \tilde{\beta}_1^{j} < \ldots < \tilde{\beta}_{b_p}^{p} < 1$$

called the parabolic weights.

REMARK 1.9. *All parabolic weights can be assigned a natural multiplicity, namely the dimension of the corresponding graded of the filtration: more precisely, the multiplicity of $\tilde{\beta}_k^p$ for any $p \in P \cup \{\infty\}$ and any $k \in \{1, \ldots, b_p\}$ is by definition*

$$\dim(F_{k-1}E_p/F_kE_p).$$

We will write

$$0 \leq \beta_1^p \leq \ldots \leq \beta_r^p < 1$$

for the parabolic weights repeated according to their multiplicities, and use this numbering of the weights throughout the whole paper instead of the one in their definition. Moreover, we write β_k^j instead of $\beta_k^{p_j}$.

REMARK 1.10. *The order of the τ_k^∞ spanning $F_m E_\infty$ in the above definition is not necessarily the same as the one in which the eigenvalues of the second-order term A at infinity appear in one group, as supposed in (1.3). However, this will not cause any confusion in the sequel, because the basis vectors at infinity in this latter order still have well-defined parabolic weights (which are then not necessarily increasing).*

DEFINITION 1.11. *A meromorphic integrable connection (E, D) with described local models and parabolic structures at the punctures will be called* parabolic integrable connection. *The* parabolic degree *of E with respect to the given parabolic structure is the real number*

$$(1.11) \qquad deg_{par}(E) = deg(E) + \sum_{j \in \{1,\ldots,n,\infty\}} \sum_{k=1}^r \beta_k^j,$$

where $deg(E)$ is the standard (algebraic geometric) degree of E, and the sum is taken over all parabolic weights for all punctures p. The slope *of the parabolic integrable connection is the real number*

$$(1.12) \qquad \mu_{par}(E) = \frac{deg_{par}(E)}{rk(E)},$$

and (E, D) is said to be parabolically stable *(resp.* semi-stable*) if for any subbundle F invariant with respect to D and endowed with the induced parabolic structure over the singularities, the inequality*

$$(1.13) \qquad \mu_{par}(F) < \mu_{par}(E)$$

(respectively $\mu_{par}(F) \leq \mu_{par}(E)$) holds. Finally, (E, D) is said to be parabolically polystable *if it is a direct sum of parabolically stable bundles that are all invariant by D and of the same slope as E.*

REMARK 1.12. *The notions of stability, semi-stability and polystability make sense for meromorphic integrable connections without a parabolic structure as well: in the corresponding definitions, one only needs to set all parabloic weights equal to 0. Notice however that by the residue theorem we have*

$$deg(E) = -\Re tr(Res(D, \infty)) - \sum_{j \in \{1,...,n\}} \Re tr(Res(D, p_j))$$

$$= \sum_{k=1}^{r} \Re \mu_k^{\infty} - \sum_{j \in \{1,...,n\}} \sum_{k=1}^{r} \Re \mu_k^{j},$$

(the change of sign coming from the fact that the eigenvalues of the residue at infinity are $-\mu_k^{\infty}$ because in the local coordinate $w = 1/z$ we have $dz/z = -dw/w$.) Therefore (1.11) is in fact equal to

$$\sum_{k=1}^{r}(\beta_k^{\infty} + \Re \mu_k^{\infty}) + \sum_{j \in \{1,...,n\}} \sum_{k=1}^{r}(\beta_k^{j} - \Re \mu_k^{j}) = \sum_{j \in \{1,...,n,\infty\}} \sum_{k=1}^{r} \gamma_k^{j},$$

where γ_k^{j} are the parabolic weights of the local system at p_j (Proposition 11.1, [5]). On the other hand, the parabolic degree of an integrable connection is always equal to 0: this follows from the Gauss-Chern formula 2.9 of [4]. Therefore, the case where the parabolic weights β_k^{j} of the integrable connection vanish is not the one where the parabolic weights γ_k^{j} of the representation of the fundamental group vanish, and where by Remark 8.2 of [6] stability reduces to irreducibility of the corresponding representation.

DEFINITION 1.13. *A Hermitian fiber metric h on E is said to be* adapted *to the parabolic structure if near the logarithmic punctures in the holomorphic bases τ_k^{j} it is mutually bounded with the diagonal model*

$$(1.14) \qquad\qquad diag(|z - p_j|^{2\beta_k^{j}})_{k=1}^{r},$$

and at infinity in the holomorphic basis τ_k^{∞} it is mutually bounded with

$$(1.15) \qquad\qquad diag(|z|^{-2\beta_k^{\infty}})_{k=1}^{r}.$$

REMARK 1.14. *In general, without the hypothesis of semisimplicity of the residue in the puctures made in Section 1.2, the local models of the metric near the punctures are more complicated than in the above definition: e.g. for the regular singularities one has to take into account*

an extra filtration induced by the nilpotent part of the residue, and add a factor $|\ln(r)|^k$ on the corresponding k-th graded, see the Synopsis of [**21**].

Here is the important existence result of the theory:

THEOREM 1.15 (Sabbah C. [**19**]; Biquard O., Boalch Ph. [**6**]). *Let (E, D) be a parabolically stable parabolic integrable connection. Then there exists a unique harmonic Hermitian metric h adapted to the parabolic structure.*

REMARK 1.16. *Actually, in the above articles this theorem is proved to hold for parabolic integrable connections having poles of arbitrary order in the punctures. On the other hand, for integrable connections with only regular singularities, it had already been shown by C. Simpson, see* [**21**].

We are now ready to describe the more general version of Nahm transform: that for parabolic integrable connections.

THEOREM 1.17. *Let (E, D) be any parabolic integrable connection on the sphere with logarithmic singularities in P as in (1.1), and a double pole (1.3) at infinity. Suppose that the eigenvalues of its polar parts μ and the parabolic weights β in the punctures satisfy the following assumptions:*

(1) *for fixed $j \in \{1, \ldots, n\}$, the complex numbers $\mu_k^j - \beta_k^j$ for $k = r_j + 1, \ldots, r$ are distinct and different from 0, the parabolic weights β_k^j for $k = 1, \ldots, r_j$ are 0 and finally $\Re\mu_k^j \notin \mathbf{Z}$ for $k = r_j + 1, \ldots, r$*

(2) *for fixed $l \in \{1, \ldots, n'\}$, the complex numbers $\mu_k^\infty - \beta_k^\infty$ for $k = 1 + a_l, \ldots, a_{l+1}$ are distinct and different from 0, and $\Re\mu_k^\infty \notin \mathbf{Z}$*

Then, in addition to the conclusions of Theorem 1.6, the transformed bundle $(\hat{E}, \hat{D})$ carries a natural parabolic structure in the punctures (that we will call transformed parabolic structure*), such that the transformed metric of the harmonic metric is adapted to it. Moreover, the set of its non-vanishing parabolic weights in $\xi_l \in \hat{P}$ is equal to the set of parabolic weights $\{\beta_{1+a_l}^\infty, \ldots, \beta_{a_{l+1}}^\infty\}$ of E at infinity, restricted to the eigenspace of A corresponding to the eigenvalue ξ_l; whereas the parabolic weights of $\hat{E}$ at infinity restricted to the eigenspace of the second-order term of $\hat{D}$ corresponding to the eigenvalue $-p_j$ are equal to the*

parabolic weights $\{\beta^j_{r_j+1}, \ldots, \beta^j_r\}$ of E at p_j. All these statements are to be understood with multiplicities.

REMARK 1.18. *Again, the conditions (1) and (2) of the theorem are generic in the parameter space of all possible eigenvalues and parabolic weights. They will regularly appear along this paper, both in analytical and geometric arguments.*

This theorem is a consequence of Theorem 1.32.

DEFINITION 1.19. *The map*

$$(1.16) \qquad\qquad \mathsf{N} : (E, D) \mapsto (\hat{E}, \hat{D})$$

described in Theorems 1.6 and 1.17 will be called Nahm transform.

Finally, as we have already mentioned, Nahm transform has an involutibility property:

THEOREM 1.20. *Let (E, D) be a parabolic integrable connection on the sphere satisfying the assumptions of Theorem 1.17. Then*

$$\mathsf{N}^2(E, D) = (-1)^*(E, D),$$

where $-1 : \mathbf{C} \to \mathbf{C}$ is the map $z \mapsto -z$, and $(-1)^*$ the induced map on fiber bundles with connection. In particular, Nahm transform is invertible.

This will be proved in Theorem 5.1, using arguments of the same type as S. K. Donaldson and P. B. Kronheimer in [**10**], namely the study of the spectral sequence of a suitable double complex.

1.5. Local model for parabolic integrable connections

We suppose in this section that near each singularity, h coincides with the diagonal models h^j and h^∞ given in Definition 1.13 (that is, without the extra $O(|z - p_j|^{2(\beta^j_k - \beta^j_{k'})})$ and $O(|z|^{-2(\beta^\infty_k - \beta^\infty_{k'})})$ factors in (1.14) and (1.15); in particular, this metric is then not harmonic). For computations, it will be useful to express the local models of the integrable connection near the singularities in some orthonormal bases. As in [**6**], we consider the orthonormal basis defined by

$$(1.17) \qquad\qquad e^j_k = |z|^{-\beta^j_k - i\Im\mu^j_k} \tau^j_k \qquad k = 1, \ldots, r$$

around p_j. The h-unitary part $(D^+)^j$ of D^j becomes

$$(1.18) \qquad\qquad (D^+)^j = \mathrm{d} + i\Re(A^j)\mathrm{d}\theta$$

where $\Re(A^j) = \frac{A^j+(A^j)^*}{2} = diag(\Re\mu_k^j)_{k=1,\dots,r}$ (and $\Im(A^j) = \frac{A^j-(A^j)^*}{2i} = diag(\Im\mu_k^j)_{k=1,\dots,r}$) stands for the real (imaginary) part of A^j, and r and θ are local polar coordinates around p_j such that we have $z - p_j = re^{i\theta}$. For the self-adjoint part Φ^j of D^j in this basis we get:

$$\Phi^j = \frac{A^j}{2}\frac{\mathrm{d}z}{z-p_j} + \frac{(A^j)^*}{2}\frac{\mathrm{d}\bar{z}}{\bar{z}-\bar{p}_j} - \beta^j\frac{\mathrm{d}r}{r}$$

$$(1.19) \qquad\qquad = [\Re(A^j) - \beta^j]\frac{\mathrm{d}r}{r} - \Im(A^j)\mathrm{d}\theta,$$

where $\beta^j = diag(\beta_k^j)_{k=1,\dots,r}$. These together imply that with respect to this basis, the model for the operator D in polar coordinates is

$$(1.20) \qquad\qquad D^j = \mathrm{d} + iA^j\mathrm{d}\theta + [\Re(A^j) - \beta^j]\frac{\mathrm{d}r}{r}.$$

In an analogous way, in the orthonormal basis $\{e_k^\infty\}_{k=1,\dots,r}$ given by

$$(1.21) \qquad\qquad e_k^\infty = |z|^{\beta_k^\infty+i\Im\mu_k^\infty}\exp\left[(\xi_k z - \bar{\xi}_k\bar{z})/2\right]\tau_k^\infty$$

near infinity the unitary part of the model connection D^∞ is given by

$$(D^+)^\infty = \mathrm{d} + i\Re(C)\mathrm{d}\theta,$$

where we have put again $\Re(C) = \frac{C+C^*}{2} = diag(\Re\mu_k^\infty)_{k=1,\dots,r}$ and $z = re^{i\theta}$. Moreover, putting $\Re(zA) = diag\Re(\{z\xi_l, m_l\})_{l=1,\dots n'}$ and $\Im(zA) = diag\Im(\{z\xi_l, m_l\})_{l=1,\dots n'}$, the self-adjoint part of D^∞ has the form

$$\Phi^\infty = \frac{1}{2}\left(A + \frac{C}{z}\right)\mathrm{d}z + \frac{1}{2}\left(A^* + \frac{C^*}{\bar{z}}\right)\mathrm{d}\bar{z} + \beta^\infty\frac{\mathrm{d}r}{r}$$

$$(1.22) \qquad\qquad = [\Re(zA + C) + \beta^\infty]\frac{\mathrm{d}r}{r} + \Im(zA + C)\mathrm{d}\theta$$

(the inversion of the sign of β comes from the fact that if we make a co-ordinate change $w = 1/z$, $|w| = \rho = 1/r = 1/|z|$, then $\mathrm{d}\rho/\rho = -\mathrm{d}r/r$). Remark that in these expressions the terms in $\mathrm{d}\theta, \mathrm{d}r/r, \mathrm{d}z/z, \mathrm{d}\bar{z}/\bar{z}$ are of lower order then the ones in $\mathrm{d}z, \mathrm{d}\bar{z}, z\mathrm{d}r/r, z\mathrm{d}\theta$; hence the leading order term of the singular part of D in this basis is just

$$(1.23) \qquad\qquad \mathrm{d} + \frac{A}{2}\mathrm{d}z + \frac{A^*}{2}\mathrm{d}\bar{z}.$$

1.6. Higgs bundle point of view

The idea of the proofs of Theorems 1.6 and 1.17 will be to exploit the correspondence known as nonabelian Hodge theory between parabolic integrable connections on one side and parabolic Higgs bundles on the other side. Let us recall the definition of the latter notion:

DEFINITION 1.21. *A parabolic Higgs bundle is given by:*

(1) *a holomorphic bundle $\mathcal{E}$ with holomorphic structure $\bar{\partial}^{\mathcal{E}}$ over* $\mathbf{CP}^1$ *called the* holomorphic bundle underlying the Higgs bundle, *and with underlying smooth vector bundle V;*

(2) *in each point $p \in P \cup \{\infty\}$ a strictly decreasing parabolic flag*

$$V_p = F_0 V_p \supset F_1 V_p \supset \ldots \supset F_{c_p-1} V_p \supset F_{c_p} V_p = \{0\}$$

for some $1 \le c_p \le r$, with parabolic weights

$$0 \le \tilde{\alpha}_1^p < \ldots < \tilde{\alpha}_{c_p}^p < 1;$$

(3) *a $\bar{\partial}^{\mathcal{E}}$-meromorphic section $\theta \in \Omega^{1,0}(\mathbf{CP}^1, End(V))$ (called the* Higgs field*), having a simple pole at the points of P with semi-simple residue respecting the parabolic flag (that is, such that $Res(\theta, p_j)$ leaves $F_k V_{p_j}$ invariant for each $p_j \in P$ and all $0 \le k \le c_p$), and a second-order pole at infinity, such that there exists a holomorphic basis of $\mathcal{E}$ near infinity compatible with the parabolic structure in which the residue and second-order term are both diagonal.*

Again, we write

$$0 \le \alpha_1^p \le \ldots \le \alpha_r^p < 1$$

for the parabolic weights repeated according to their multiplicities

$$\dim(F_{k-1} V_p / F_k V_p),$$

and we shorten $\alpha_k^{p_j}$ to α_k^j. Finally, we set

(1.24) $$D'' = \bar{\partial}^{\mathcal{E}} + \theta,$$

that we call the D''-operator associated to the Higgs bundle.

The notions of parabolic degree, slope and (poly/semi-)stability of parabolic Higgs bundles are defined analogously to the case of integrable connections, see Definition 1.11. Theorem 6.1 of [**6**] then says the following.

THEOREM 1.22 (Biquard O.-Boalch Ph., 2004). *There exists an isomorphism between the moduli space of parabolically stable rank r integrable connections with fixed diagonal polar part and parabolic structures up to complex holomorphic gauge transformations respecting the parabolic flags, and the moduli space of parabolically stable rank r Higgs bundles with fixed diagonal polar part and parabolic structures up to complex holomorphic gauge transformations respecting the parabolic flags.*

REMARK 1.23. *Actually, this is a consequence of the existence of a harmonic metric (Theorem 1.15), and hence also proved for parabolic integrable connections with poles of arbitrary fixed order and diagonal polar part in the punctures and parabolic Higgs bundles with poles of the same order with diagonal polar part.*

The transition from integrable connections to Higgs bundles is given as follows: first, the underlying smooth vector bundle of the integrable connection and the Higgs bundle are the same. Furthermore, suppose h is the harmonic metric, consider the decomposition (1.8) of the integrable connection into its unitary and self-adjoint part, and decompose the terms further according to bidegree

$$\text{(1.25)} \qquad D^+ = (D^+)^{1,0} + (D^+)^{0,1}$$
$$\Phi = \Phi^{1,0} + \Phi^{0,1}.$$

The partial connection $(D^+)^{0,1}$ defines then the holomorphic structure of $\mathcal{E}$, and $\Phi^{1,0}$ will be the Higgs field θ. The D''-operator is of course $(D^+)^{0,1} + \Phi^{1,0}$. Harmonicity of the metric implies that θ is holomorphic.

The transition in the other direction is also established using a privileged metric.

DEFINITION 1.24. *Let $(\mathcal{E}, \theta)$ be a Higgs bundle. We say that h is a Hermitian-Einstein metric for $(\mathcal{E}, \theta)$ if, denoting by D_h^+ the Chern connection (the unique h-unitary connection compatible with $\bar{\partial}^{\mathcal{E}}$), by $F_{D_h^+}$ its curvature, and by θ_h^* the adjoint of θ with respect to h, then these objects satisfy the real Hitchin equation*

$$F_{D_h^+} + [\theta, \theta_h^*] = 0,$$

where $[.,.]$ stands for graded commutator of forms.

Let $(\mathcal{E}, \theta)$ be a parabolically stable parabolic Higgs bundle. By [6], there exists a unique Hermitian-Einstein metric h adapted to the parabolic structure. The connection

$$(1.26) \qquad\qquad D = D_h^+ + (\theta + \theta_h^*)$$

on V is then integrable, and h is the corresponding harmonic metric adapted to the parabolic structure. In what follows, in order to simplify notations, we are often going to omit the subscript h in the notation of the Chern connection and adjoints.

Let now (E, D) be a parabolically stable parabolic integrable connection and $(\mathcal{E}, \theta)$ the associated parabolic Higgs bundle. An important result we will be constantly using is the following

THEOREM 1.25 (Simpson C. [21]). *Suppose the metric h is harmonic. Then, with the previous notations, the Laplace operators $\Delta_D = DD^* + D^*D$ and $\Delta_{D''} = D''(D'')^* + (D'')^*D''$ satisfy*

$$\Delta_D = 2\Delta_{D''}.$$

In particular, their domain and kernel coincide.

1.7. Local model for Higgs bundles

In this section, we give the eigenvalues of the residue of the Higgs field and the parabolic weights of the Higgs bundle in the punctures that correspond to those of the integrable connection via the Theorem 1.22. To obtain local models for the operators in this setup, suppose again that near p_j the metric h coincides with the diagonal model h^j given by (1.14) (without the correcting $O(|z - p_j|^{2(\beta_k^j - \beta_{k'}^j)})$ term; in particular, it does not satisfy Hitchin's equation). Then, according to [6], in the local $\bar{\partial}^{\mathcal{E}}$-holomorphic trivialisation

$$(1.27) \qquad \sigma_k^j = |z - p_j|^{\Re \mu_k^j} \frac{e_k^j}{(z - p_j)^{[\Re \mu_k^j]}} \qquad\qquad (k = 1, \ldots, r)$$

around p_j, the Higgs field is equal up to a perturbation term to the model Higgs field given by

$$
\begin{aligned}
\theta^j &= \frac{A^j - \beta^j}{2} \frac{\mathrm{d}z}{z - p_j} \\
&= diag\left(\frac{\mu_k^j - \beta_k^j}{2} \frac{\mathrm{d}z}{z - p_j} \right)_{k=1,\ldots,r} \\
&= diag\left(\lambda_k^j \frac{\mathrm{d}z}{z - p_j} \right)_{k=1,\ldots,r},
\end{aligned}
$$

(1.28)

where we have put $\lambda_k^j = (\mu_k^j - \beta_k^j)/2$. Moreover, in the same trivialisation, the parabolic weights are

(1.29)
$$
\alpha_k^j = \Re(\mu_k^j) - [\Re(\mu_k^j)],
$$

where $[.]$ denotes integer part.

REMARK 1.26. *In fact, this formula is not completely correct, because the α_k^j defined by it are not necessarily in increasing order, although they should be by definition. One should instead write the same formula for $\alpha_{s(k)}^j$, where s is a permutation of $\{1, \ldots, r\}$. However, in the sequel we discard this minor technical detail for the sake of simplicity of the notation.*

REMARK 1.27. *Since the gauge transformations between the bases $\{\tau_k^j\}_{k=1,\ldots,r}$ and $\{\sigma_k^j\}_{k=1,\ldots,r}$ are just multiplications by some functions (in particular diagonal matrices), it follows that the smooth subbundle spanned by the sections $\{\sigma_k^j\}_{k=r_j+1,\ldots,r}$ is the same as the one spanned by $\{\tau_k^j\}_{k=r_j+1,\ldots,r}$, which is by definition the underlying smooth vector bundle of E_{sing}^j; and similarly, the subbundle spanned by $\{\sigma_k^j\}_{k=1,\ldots,r_j}$ is equal to the underlying smooth bundle of E_{reg}^j. The same remark also holds for the bases $\{e_k^j\}$ instead of $\{\sigma_k^j\}$. In particular, the residue of the model Higgs field θ^j in the point $p_j \in P$ belongs to $End(E_{sing}^j|_{p_j})$.*

Near infinity, the situation is slightly different: for $h = h^\infty$ the diagonal model, in the local $\bar{\partial}^{\mathcal{E}}$-holomorphic frame

(1.30)
$$
\sigma_k^\infty = |z|^{-\Re\mu_k^\infty} z^{[\Re\mu_k^\infty]} e_k^\infty \qquad (k = 1, \ldots, r)
$$

the Higgs field is equal up to a perturbation term to the model Higgs field given by

$$\theta^\infty = \frac{1}{2}A\mathrm{d}z + \frac{\mu^\infty - \beta^\infty}{2}\frac{\mathrm{d}z}{z}$$

$$(1.31) \qquad = \left(\frac{1}{2}diag(\{\xi_l, m_l\})_{l=1,\dots,n'} + \frac{1}{z}diag(\lambda_k^\infty)_{k=1,\dots,r}\right)\mathrm{d}z,$$

where we have put again $\lambda_k^\infty = (\mu_k^\infty - \beta_k^\infty)/2$, with parabolic weights being, as in the case of simple poles,

$$(1.32) \qquad \alpha_k^\infty = \Re(\mu_k^\infty) - [\Re(\mu_k^\infty)].$$

From these data, as above, one can form the model D''-operator

$$(1.33) \qquad (D'')^j = \bar{\partial}^{\mathcal{E}} + \theta^j \qquad (j \in \{1, \dots n, \infty\}).$$

Notice that since we considered holomorphic trivialisations of $\mathcal{E}^j$, the partial connection part of the model coincides with the usual $\bar{\partial}$-operator.

We are now ready to write out the assumptions made in Theorem 1.17 on the parameters of the integrable connection, translated to those of the Higgs bundle:

HYPOTHESIS 1.28. *We suppose that $(\mathcal{E}, \theta)$ is a parabolically stable Higgs bundle with diagonal polar part of the Higgs field in some local holomorphic frame near each puncture, satisfying the properties*
   (1) *for fixed $j \in \{1, \dots, n\}$ the residues λ_k^j for $k \in \{r_j + 1, \dots, r\}$ are non-vanishing and distinct, λ_k^j vanish for $k = 1, \dots, r_j$ and finally $\alpha_k^j \neq 0$ if and only if $\lambda_k^j \neq 0$;*
   (2) *for fixed $l \in \{1, \dots, n'\}$ the complex numbers λ_k^∞ for $k \in \{1 + a_l, \dots a_{l+1}\}$ are non-vanishing and distinct, and $\alpha_k^\infty \neq 0$.*

Diagonality of the polar parts has already been assumed when writing the local models (1.28) and (1.31). The first condition says that no parabolic weight and no eigenvalue of the residue of θ vanishes on the singular component at any singularity, and that on the singular component near a puncture all eigenvalues are different; whereas the eigenvalues of the residue and parabolic weights vanish on the regular component. One more way to say the same thing is: for all $j \in \{1, \dots n\}$, the residue of θ defines an automorphism of $E_{sing}^j|_{p_j}$, and the parabolic weights corresponding to the holomorphic trivialisation (1.27) are non-vanishing exactly on this subspace. The second one imposes that on the eigenspace corresponding to a fixed eigenvalue

of the second-order term at infinity, all the eigenvalues of the residue be non-vanishing and distinct, furthermore that no parabolic weight vanish at infinity. Note that these conditions are generic among all possible choices of singularity parameters.

1.8. The transformation of the Higgs bundle

Let $(\mathcal{E}, \theta)$ be a parabolic Higgs bundle and $\xi \in \hat{\mathbf{C}} \setminus \hat{P}$ a parameter. The natural deformation of the Higgs field is

$$(1.34) \qquad \theta_\xi = \theta - \frac{\xi}{2}\mathrm{d}z$$

with fixed underlying holomorphic bundle $\mathcal{E}$. It is clear that θ_ξ is then also holomorphic with respect to $\bar{\partial}^{\mathcal{E}}$ with the same local models at the logarithmic punctures as θ, but its local model near infinity is different. If moreover a Hermitian metric is fixed, then we also have

$$\theta_\xi^* = \theta^* - \frac{\bar{\xi}}{2}\mathrm{d}\bar{z}.$$

Therefore, the integrable connection corresponding to the deformed Higgs bundle is given by

$$(1.35) \qquad D_\xi^H = D - \frac{\xi}{2}\mathrm{d}z - \frac{\bar{\xi}}{2}\mathrm{d}\bar{z},$$

and the crucial observation is that via the unitary gauge transformation

$$(1.36) \qquad \exp[(\bar{\xi}\bar{z} - \xi z)/2]$$

on $\mathbf{C}$ this is equivalent to the deformation (1.4). The self-dual part of this deformation is

$$(1.37) \qquad \Phi_\xi^H = \Phi - \frac{\xi}{2}\mathrm{d}z - \frac{\bar{\xi}}{2}\mathrm{d}\bar{z},$$

the same deformation as in (1.6). Therefore it will not make any confusion to refer to Φ_ξ without mentioning the adopted point of view; consequently, we drop the corresponding upper indices. The connection defined by (1.35) is still flat, but the underlying holomorphic structure is different from the one of D (because of the term in $\mathrm{d}\bar{z}$). Notice also that the gauge transformation (1.36) between these deformations has an exponential singularity at infinity. Denote by $\mathcal{C}_\xi^H$ the elliptic complex

$$(1.38) \qquad \Omega^0 \otimes E \xrightarrow{D_\xi^H} \Omega^1 \otimes E \xrightarrow{D_\xi^H} \Omega^2 \otimes E.$$

DEFINITION 1.29. *The smooth vector bundle $\hat{V}$ underlying the transformed Higgs bundle is the vector bundle whose fiber over $\xi \in \hat{\mathbf{C}} \setminus \hat{P}$ is the first L^2-cohomology $L^2 H^1(\mathcal{C}_\xi^H)$ of $\mathcal{C}_\xi^H$.*

In Proposition 4.2 we prove that these vector spaces indeed define a finite rank smooth bundle. Furthermore, by Theorem 2.21, any class in $L^2 H^1(\mathcal{C}_\xi^H)$ admits a unique D_ξ^H-harmonic representative.

DEFINITION 1.30. *The transformed holomorphic structure on $\hat{V}$ is the one induced by the orthogonal projection $\bar{\partial}^{\mathcal{E}}$ of the trivial partial connection with respect to the variable ξ on the trivial L^2-bundle over $\widehat{\mathbf{CP}}^1$ to D_ξ^H-harmonic 1-forms. The transformed Higgs field is multiplication by $-z d\xi/2$ followed by projection onto harmonic 1-forms. Finally, the transformed Hermitian metric is the L^2-metric of the harmonic representative.*

By virtue of Theorems 2.21 and 1.25, the transformed smooth bundle $\hat{V}$ can also be computed in this case as the first cohomology of the elliptic complex $\mathcal{C}_\xi''$ given by:

$$\Omega^0 \otimes E \xrightarrow{D_\xi''} \Omega^1 \otimes E \xrightarrow{D_\xi''} \Omega^2 \otimes E,$$

where the maps are the corresponding deformations of (1.24) in the Higgs-bundle point of view. Explicitly, D_ξ'' reads

$$(D_\xi^H)'' = \bar{\partial}^{\mathcal{E}} + \theta_\xi.$$

We use this description of the transformed bundle in Section 4.2 to show the statement of Theorem 1.6 on the transformed metric:

THEOREM 1.31. *If the original metric harmonic then the same thing holds for the transformed metric.*

For this purpose, we prove in fact that the candidate Higgs field $\hat{\theta}$ corresponding to $\hat{D}$ and $\hat{h}$ is meromorphic with respect to the transformed holomorphic structure.

Furthermore, in this interpretation, the remaining part of Theorems 1.6 and 1.17 can be written:

THEOREM 1.32. *Suppose $(\mathcal{E}, \theta)$ is a parabolic Higgs bundle with logarithmic singularities in the points of P and a double pole at infinity, as described in Section 1.6, such that its singularity parameters satisfy Hypothesis 1.28. Then the transformed Higgs bundle $(\bar{\partial}^{\hat{\mathcal{E}}}, \hat{\theta})$ is of the*

same type (that is, it has a finite number of logarithmic singularities in points of $\hat{\mathbf{C}}$ and a double pole at infinity, with a parabolic structure in these points). Furthermore, its topological and singularity parameters are as follows:

(1) *the rank of $\hat{\mathcal{E}}$ is the sum (1.10) of the ranks of the residues of θ in P*

(2) *its degree is the same as that of $\mathcal{E}$*

(3) *the logarithmic singularities are located in the set $\hat{P}$, and for all $l \in \{1, \ldots, n'\}$ the rank of the transformed Higgs field in the point ξ_l is equal to the multiplicity m_l of the eigenvalue ξ_l of A*

(4) *the set of non-vanishing eigenvalues of the residue of $\hat{\theta}$ in the point ξ_l is $\{-\lambda_{1+a_l}^{\infty}, \ldots, -\lambda_{a_{l+1}}^{\infty}\}$, where $\{\lambda_{a_l+1}^{\infty}, \ldots, \lambda_{a_{l+1}}^{\infty}\}$ are the eigenvalues of the residue of the original Higgs field θ at infinity, restricted to the eigenspace of A corresponding to the eigenvalue ξ_l*

(5) *the non-vanishing parabolic weights of $\hat{\mathcal{E}}$ in ξ_l is the set of parabolic weights $\{\alpha_{1+a_l}^{\infty}, \ldots, \alpha_{a_{l+1}}^{\infty}\}$ of $\mathcal{E}$ at infinity, restricted to the same subspace*

(6) *the eigenvalues of the second-order term of $\hat{\theta}$ at infinity are $\{-p_1/2, \ldots, -p_n/2\}$, and the multiplicity of $-p_j/2$ is equal to the rank $r - r_j$ of the residue of θ in p_j*

(7) *on the eigenspace corresponding to $-p_j/2$ of the second-order term at infinity, the eigenvalues of the residue of $\hat{\theta}$ are $\{-\lambda_{r_j+1}^{j}, \ldots, -\lambda_r^{j}\}$*

(8) *the parabolic weights on the same eigenspace at infinity are the parabolic weights $\{\alpha_{r_j+1}^{j}, \ldots, \alpha_r^{j}\}$ of $\mathcal{E}$ at p_j*

The proof of this theorem is the object of Chapter 4.

CHAPTER 2

Analysis of the Dirac operator

In this chapter, we study the analytical theory needed for our construction along the lines of Donaldson-Kronheimer [10], Jardim [15] and others. First, in Section 2.1 we define spinor spaces and Dirac operators that permit us to study the problem. We also define a suitable functional space H^1 and state a Fredholm theorem valid for all deformations of the initial connection. Then it is natural to define the fibers of the transformed bundle as the cokernel of the deformed Dirac operator. The Fredholm theorem is then proved in Section 2.2. In Section 2.3, we carry out an identification of this cokernel with the first L^2-cohomology $L^2H^1(D_\xi^{int})$ of the complex $\mathcal{C}_\xi^{int}$ given in (1.7), similar in vein to the Hodge isomorphism between the cokernel of the operator $d + d^*$ on a compact manifold and the L^2-cohomology of the operator d. However, since the manifold we are working on is non-compact, in proving these results we need a careful study of the singularities.

In all what follows, we fix a parabolic integrable connection with adapted metric (E, D, h) and choose to study the analytic properties of the deformation from the point of view of integrable connections, hence we set for simplicity $D_\xi = D_\xi^{int}$ until further notification.

2.1. Statement of the Fredholm theorem

DEFINITION 2.1. *The positive and negative* spinor bundles *are the vector bundles over* $\mathbf{C} \smallsetminus P$ *given by*

$$S^+ = \Lambda^0 T^*(\mathbf{C} \smallsetminus P) \oplus \Lambda^2 T^*(\mathbf{C} \smallsetminus P) \qquad S^- = \Lambda^1 T^*(\mathbf{C} \smallsetminus P)$$

Recall that we have defined $\hat{P}$ as the set $\{\xi_1, \ldots, \xi_r\}$ of all eigenvalues of the second order term of D at infinity.

DEFINITION 2.2. *For* $\xi \in \hat{\mathbf{C}} \smallsetminus \hat{P}$ *the* Dirac operator *is the first-order differential operator*

$$\slashed{\partial}_\xi = D_\xi - D_\xi^* : \Gamma(S^+ \otimes E) \longrightarrow \Gamma(S^- \otimes E)$$

where Γ is used to denote smooth sections with compact support in $\mathbf{C} \smallsetminus P$. *Its formal adjoint*

$$\not{D}_\xi^* = D_\xi^* - D_\xi : \Gamma(S^- \otimes E) \longrightarrow \Gamma(S^+ \otimes E),$$

is called the adjoint Dirac operator.

For any $\xi \in \hat{\mathbf{C}}$ let us introduce the following norm on sections f of $S^+ \otimes E$:

$$\tag{2.1} \|f\|_{H_\xi^1}^2 = \int_{\mathbf{C}} |f|^2 + |\nabla_\xi^+ f|^2 + |\Phi_\xi \otimes f|^2,$$

where ∇_ξ^+ and Φ_ξ are defined in (1.5) and (1.6). Here and in all what follows, we integrate with respect to the Euclidean volume form $|\mathrm{d}z|^2$, and $|x|^2$ denotes $h(x,x)$, unless the contrary is explicitly stated. Our convention is furthermore to write (x,y) for $h(x,y)$, and for sections x and y, we write $\langle x, y \rangle$ instead of $\int_{\mathbf{C}}(x,y)$.

Define the space of sections

$$\tag{2.2} H_\xi^1(S^+ \otimes E) = \{f \in L_{loc}^2(S^+ \otimes E) : \|f\|_{H_\xi^1} < \infty\},$$

where in L^2 we refer to the metric h on the fibers. We will often write H_ξ^1 instead of $H_\xi^1(S^+ \otimes E)$. As we will see by the end of this chapter, this is the appropriate space to regard the Dirac operators. First we establish the simple

LEMMA 2.3. *The norm* $\|.\|_{H_\xi^1}$ *depends (up to equivalence of norms) neither on* $\xi \in \hat{\mathbf{C}}$, *nor on the particular connection* D *having behavior as in (1.1) and (1.3).*

PROOF. We begin by showing that the norm is independent of ξ. In order to simplify notations, we let H^1 stand for H_0^1 from now on. It is obviously sufficient to prove that for an arbitrary $\xi \in \hat{\mathbf{C}}$, the H_ξ^1-norm is equivalent to the H^1-norm. From the point-wise identity

$$|\Phi_\xi \otimes f| = 2|\theta_\xi \otimes f| = 2|\theta_\xi^* \otimes f|,$$

and the point-wise estimation

$$\tag{2.3} |\nabla_\xi^+ f| \le |\nabla^+ f| + 2|\xi||f|,$$

one sees that for any section $f = (f_0, f_2) \in \Gamma(S^+ \otimes E)$ the estimates

$$\|f\|_{H_\xi^1}^2 \le (1 + 8|\xi|^2)\|f\|_{H^1}^2$$

and

$$\|f\|_{H^1}^2 \le (1 + 8|\xi|^2)\|f\|_{H_\xi^1}^2$$

hold; the first statement of the Lemma follows at once.

Now we show independence of the particular connection D with right singularity behavior. Introduce the model norm

$$(2.4) \qquad \|f\|^2_{H^1_{mod}(\Delta(p_j,\varepsilon))} = \int_{\Delta(p_j,\varepsilon)} |f|^2 + |(D^+)^j f|^2 + |\Phi^j f|^2$$

around points of P and the model norm

$$(2.5) \quad \|f\|^2_{H^1_{mod}(\mathbf{C}\smallsetminus\Delta(0,R))} = \int_{\mathbf{C}\smallsetminus\Delta(0,R)} |f|^2 + |(D^+)^\infty f|^2 + |\Phi^\infty f|^2$$

near infinity. Then it is sufficient to prove the following:

CLAIM 2.4. *If $\varepsilon > 0$ is chosen sufficiently small and $R > 0$ sufficiently large, then for any smooth section $f \in H^1$ we have*

$$(2.6) \qquad c\|f^j\|^2_{H^1(\Delta(p_j,\varepsilon))} < \|f^j\|^2_{H^1_{mod}(\Delta(p_j,\varepsilon))} < C\|f^j\|^2_{H^1(\Delta(p_j,\varepsilon))}$$

and similarly

$$(2.7) c\|f^j\|^2_{H^1(\mathbf{C}\smallsetminus\Delta(0,R))} < \|f^j\|^2_{H^1_{mod}(\mathbf{C}\smallsetminus\Delta(0,R))} < C\|f^j\|^2_{H^1(\mathbf{C}\smallsetminus\Delta(0,R))}$$

with some constants $0 < c < C$ independent of f.

PROOF. Consider first the case of $p_j \in P$. Decompose $f^j = f^j_{reg} + f^j_{sing}$ corresponding to the splitting (1.2). Write also

$$(2.8) \qquad\qquad f^j_{reg} = \sum_{k=1}^{r_j} \phi^j_k e^j_k$$

$$(2.9) \qquad\qquad f^j_{sing} = \sum_{k=r_j+1}^{r} \phi^j_k e^j_k$$

with respect to the orthonormal basis $\{e^j_k\}$ introduced in (1.17), where the ϕ^j_k are functions. Formulas (1.18) and (1.19) and Hypothesis 1.28 imply that (2.4) is equivalent to the weighted Sobolev space of sections satisfying

$$(2.10) \qquad \sum_{k=1}^{r_j} \int_{\Delta(p_j,\varepsilon)} |\phi^j_k|^2 + |\mathrm{d}\phi^j_k|^2$$

$$+ \sum_{k=r_j+1}^{r} \int_{\Delta(p_j,\varepsilon)} \left| \frac{\phi^j_k}{r} \right|^2 + |\mathrm{d}\phi^j_k|^2 < \infty,$$

where d stands for the trivial connection on functions. Notice that we only add weights on the singular component. By [21], Theorem 1 it

follows that in $\Delta(p_j, \varepsilon)$ the difference between $(D^+)^j$ and D^+ is

$$(2.11) \qquad\qquad\qquad a^j = O(r^{-1+\delta})$$

for some $\delta > 0$, and the same estimation holds for the difference between Φ^j and Φ. It is then immediate that for any $c > 0$, the estimation

$$\int_{\Delta(p_j,\varepsilon)} \left| \frac{\phi_k^j}{r} \right|^2 > c \int_{\Delta(p_j,\varepsilon)} |a^j \phi_k^j|^2$$

holds for $k = r_j + 1, \ldots, r$ and for $\varepsilon > 0$ sufficiently small. We therefore have (2.6) for f_{sing}. On the other hand, for a function g defined in $\Delta(0, 1)$ and for $\delta > 0$ fixed, from the claim in the proof of Theorem 5.4 in [6] we have

$$\int_{\Delta(p_j,1)} |r^{-1+\delta}g|^2 \le c \left(\int_{\Delta(p_j,1)} |\mathrm{d}g|^2 + \int_{\Delta(p_j,1) \setminus \Delta(p_j,1/2)} |g|^2 \right).$$

Rescaling this inequality to the disk $\Delta(p_j, \varepsilon)$ one easily checks that it implies

$$\varepsilon^{-2\delta} \int_{\Delta(p_j,\varepsilon)} |r^{-1+\delta}g|^2$$

$$(2.12) \qquad\qquad \le c \left(\int_{\Delta(p_j,\varepsilon)} |\mathrm{d}g|^2 + \varepsilon^{-2} \int_{\Delta(p_j,\varepsilon) \setminus \Delta(p_j,\varepsilon/2)} |g|^2 \right).$$

Choosing ε sufficiently small, applying this to ϕ_k^j for $k = 1, \ldots, r_j$, and recalling that on the regular component $(D^+)^j$ is the trivial connection d and $\Phi^j = 0$, we obtain (2.6) for f_{reg} as well. This establishes the equivalence of the norms $\|.\|_{H^1_{mod}}^2$ and $\|.\|_{H^1}^2$ around a finite singularity.

Around infinity, by [6] Lemma 4.6 the difference between $(D^+)^\infty$ and D^+ is bounded above by a term

$$(2.13) \qquad\qquad\qquad a^\infty = O(r^{-1-\delta})$$

for some $\delta > 0$, and again the same holds for $\Phi^\infty - \Phi$. The equivalence (2.7) follows immediately from the estimation

$$|r^{-1-\delta}f| \le c|f|$$

for any $c > 0$, whenever $r > R$ with R sufficiently large.   □

This then finishes the proof of Lemma 2.3 as well.

□

From the previous discussion, we bring out as consequence:

COROLLARY 2.5. *The Hilbert space $H^1(E)$ is the set of sections $f \in L^{2,1}_{loc}(E)$ such that near a logarithmic singularity p_j, in the decompositions (2.8) and (2.9) we have $\phi^j_k \in L^{2,1}$ for $k = 1, \ldots, r_j$ and $\phi^j_k/r, d\phi^j_k \in L^2$ for $k = r_j + 1, \ldots, r$; whereas at infinity, the coordinates ϕ^∞_k of f in the basis (1.21) are $L^{2,1}$; equipped with the norm*

$$\int_{\mathbf{C} \smallsetminus \cup_j \Delta(p_j, \varepsilon)} |f|^2 + |\nabla f|^2$$

$$+ \sum_{j=1}^{n} \left\{ \sum_{k=1}^{r_j} \int_{\Delta(p_j, \varepsilon)} |\phi^j_k|^2 + |d\phi^j_k|^2 + \sum_{k=r_j+1}^{r} \int_{\Delta(p_j, \varepsilon)} \left| \frac{\phi^j_k}{r} \right|^2 + |d\phi^j_k|^2 \right\}$$

The same result holds for sections of $\Omega^2 \otimes E$, coordinates being expressed in the basis $dz \wedge d\bar{z}$.

PROOF. For sections of Ω^0, this follows from Claim 2.4, (2.10) and

$$|\Phi \otimes f| \leq K|f|.$$

We then obtain the case of Ω^2 by duality. $\qquad\square$

We now come back to the analysis of the Dirac operator. From the definitions of $H^1(S^+ \otimes E)$ and $\slashed{\partial}_\xi$ we see that this latter admits a bounded extension

(2.14) $$\slashed{\partial}_\xi : H^1(S^+ \otimes E) \longrightarrow L^2(S^- \otimes E).$$

We are now able to announce the first main result of this chapter:

THEOREM 2.6. *The operator (2.14) is Fredholm; if h is harmonic, its kernel vanishes.*

COROLLARY 2.7. *The bundle over $\hat{C} \smallsetminus \hat{P}$ whose fiber over ξ is the cokernel of (2.14) is a smooth vector bundle.*

PROOF. We recall the well-known fact that the index of a continuous family of Fredholm operators is constant. On the other hand, if the kernel of a Fredholm operator vanishes, then its index is equal to the opposite of the dimension of its cokernel. It then follows immediately from the Fredholm theorem that if the metric is harmonic, then the dimension of the cokernel of the operator $\slashed{\partial}_\xi$ is a finite constant independent of ξ. Moreover, by standard implicit function theorem arguments in Hilbert space it follows that the cokernels of these Dirac operators in $L^2(S^- \otimes E)$ vary smoothly with ξ. $\qquad\square$

Therefore, we may set the following.

DEFINITION 2.8. *The* transformed vector bundle $\hat{E}$ *of* (E, D, h) *of a singular integrable connection with harmonic metric is the smooth vector bundle over* $\hat{\mathbf{C}} \setminus \hat{P}$ *whose fiber over* ξ *is the finite-dimensional vector space* $\hat{E}_\xi = coKer(\not{\partial}_\xi) \subset L^2(S^- \otimes E)$.

In the remaining of this section, we prove vanishing of the kernel. The proof of the first statement of Theorem 2.6 is left for the next section. For the rest of the discussion in this section, we drop the index ξ.

LEMMA 2.9. *The subspaces* $Im(\not{\partial}|_{H^1(\Omega^0)})$ *and* $Im(\not{\partial}|_{H^1(\Omega^2)})$ *of* $L^2(\Omega^1)$ *are orthogonal.*

PROOF. Let $f_0 \in H^1(\Omega^0)$ and $f_2 = g \mathrm{d}z \wedge \mathrm{d}\bar{z} \in H^1(\Omega^2)$. Suppose first that f_0 is smooth and supported on a compact subset of $\mathbf{C}$, and such that near any singularity $p_j \in P$ its singular part is supported away from p_j. Then in a neighborhood of any p_j in a holomorphic basis $Df_0 = (\mathrm{d} + a)f$ for some bounded section $a \in \mathbf{\Omega}^1(End(\mathcal{E}))$, and so we have by partial integration

$$(2.15) \qquad \int_{\mathbf{C} \setminus P} (Df_0, D^* f_2) = \int_{\mathbf{C} \setminus P} (D^2 f_0, f_2) = 0,$$

since D is flat. Therefore, in order to finish the proof it is sufficient to show the following:

CLAIM 2.10. *The set of compactly supported smooth sections of* $S^+ \otimes E$ *on* $\mathbf{C}$ *with singular part compactly supported away from any singularity is dense in* H^1.

PROOF. It is sufficient to show the statement for Ω^0, the case of Ω^2 being analogous. First we concentrate on infinity. Let $f \in H^1(E)$, and define cut-off functions $\rho_R(r)$ supported in $[0, 2R]$ and equal to 1 on $[0, R]$, such that ρ'_R is supported in $[R, 2R]$ with

$$max|\rho'_R| \leq 2/R.$$

Then we need to check that

$$\rho_R(r)f \longrightarrow f$$

in $H^1(E)$ as $R \to \infty$. In view of Corollary 2.5, this boils down to the classical calculations

$$\|(1 - \rho_R(r))f\| \leq \int_{R \leq r} |f|^2$$

and

$$\|\nabla^+((1-\rho_R(r))f)\| \leq \int_{R \leq r \leq 2R} |\rho_R'(r)|^2 |f|^2 + K \int_{R \leq r} |\nabla^+ f|^2$$

$$\leq K' \int_{R \leq r \leq 2R} |f|^2 + K \int_{R \leq r} |\nabla^+ f|^2,$$

where K, K' are constants independent of R and f.

Next, let us consider a logarithmic singularity p_j, and define cut-off functions ρ_ε supported in $[0, \varepsilon]$, equal to 1 on $[0, \varepsilon/2]$, and such that

$$\max |\rho_\varepsilon'| \leq \frac{4}{\varepsilon}.$$

We need to show that

$$(1 - \rho_\varepsilon) f^{sing} \longrightarrow f^{sing}$$

in $H^1(E)$ as $\varepsilon \to 0$. One sees that

$$\int_{\mathbf{C}} |\rho_\varepsilon f^{sing}|^2 \leq \int_{r < \varepsilon} |f^{sing}|^2 \to 0,$$

since $f^{sing} \in L^2$. In the same way,

$$\int_{\mathbf{C}} \frac{|\rho_\varepsilon f^{sing}|^2}{r^2} \leq \int_{r < \varepsilon} \frac{|f^{sing}|^2}{r^2} \to 0,$$

since $f^{sing}/r \in L^2$. Finally, we also see that

$$\int_{\mathbf{C}} |\nabla^+(\rho_\varepsilon f^{sing})|^2 \leq \frac{16}{\varepsilon^2} \int_{\varepsilon/2 < r < \varepsilon} |f^{sing}|^2 + \int_{r < \varepsilon} |\nabla^+ f^{sing}|^2$$

$$\leq \int_{\varepsilon/2 < r < \varepsilon} \frac{16|f^{sing}|^2}{r^2} + \int_{r < \varepsilon} |\nabla^+ f^{sing}|^2$$

and all of these expressions converge to zero as well. $\qquad \square$

Applying the claim to approximate f_0 and f_2 in H^1 by sections with compactly supported singular component combined with (2.15), we immediately get the lemma. $\qquad \square$

Now we can come to vanishing of the kernel of (2.14): by Lemma 2.9, we have

$$Ker(\not{D}_\xi) = Ker(D_\xi|_{H^1(\Omega^0)}) \oplus Ker(D_\xi^*|_{H^1(\Omega^2)}),$$

it is therefore sufficient to prove vanishing of the kernels of D and of D^*. By duality, we only need to treat the case of D. Harmonicity of the metric implies the Weitzenböck formula:

(2.16) $$\not{D}_\xi^* \not{D}_\xi = (\nabla_\xi^+)^*(\nabla_\xi^+) + (\Phi_\xi \otimes)^* \Phi_\xi \otimes$$

(see [**5**], Thm 5.4.), which then gives by partial integration and Claim
2.10 the identity

$$(2.17) \qquad \|\dot{\partial}_\xi f\|^2_{L^2} = \|D^+_\xi f\|^2_{L^2} + \|\Phi_\xi f\|^2_{L^2}$$

for any $f \in H^1(\Omega^0)$. Suppose now that f is in the kernel of $\dot{\partial}_\xi$. Then
(2.17) implies $\Phi_\xi f = 0$, and since Φ_ξ is an isomorphism near infinity
because of the choice $\xi \notin \hat{P}$, we also have there $f = 0$. Again by
(2.17), f is covariant constant. This gives the result, since a covariant
constant section vanishing on an open set vanishes everywhere.

2.2. Proof of the Fredholm Theorem

A modification of the usual gluing argument of Fredholm-type the-
orems works in this case as well. One lets ϕ_1 be a cut-off function
supported in a compact region R outside a neighborhood of the sin-
gularities, and puts $\phi_2 = 1 - \phi_1$. Since $\dot{\partial}$ is a non-singular first-order
elliptic operator in R, elliptic theory of a compact manifold implies
that a parametrix P_1 exists for $\dot{\partial}$ in this region. Next, one considers
the problem in neighborhoods of the singularities. First, one studies
the model operators $\dot{\partial}^j = D^j + (D^j)^*$ instead of the Dirac operator
itself. There are two different ways of treating these:

(1) either one extends the functional spaces and the model Dirac
    operator onto a natural completion of the neighborhood, which
    can be either a conformal cylinder or a complex line (depend-
    ing on the form of the metric and the functional spaces), and
    defines a two-sided inverse of $\dot{\partial}^j$ on this completion
(2) or one finds directly a two-sided inverse of $\dot{\partial}^j$ on a small disk
    around the singularity, with a boundary condition verified by
    any section supported outside a neighborhood of the boundary.

Let us see how these allow to deduce the Fredholm theorem: if we
take R sufficiently large, then on the support of ϕ_2 all of these inverses
$(\dot{\partial}^j)^{-1}$ are defined. One then sets

$$P : L^2(S^- \otimes E) \longrightarrow H^1(S^+ \otimes E)$$

$$P(u) = \phi_1 P_1(\phi_1 u) + \sum_j \phi_2 (\dot{\partial}^j)^{-1}(\phi_2 u),$$

and shows that this operator is a two-sided parametrix of $\dot{\partial}$ on all
C. This can be done along classical lines, the only difference being

that near the singularities we have inverses of the local models of the operator and not inverses of the operator itself. Therefore, we proceed as follows: first, we study the local models of the Dirac operator around the singularities, and establish the isomorphisms as in (1) or in (2). Then we prove that the effect of passing to the model operators from the global ones at the singularities only amounts to adding a compact operator $H^1(S^+ \otimes E) \to L^2(S^- \otimes E)$, which then gives the theorem.

2.2.1. Logarithmic singularities. Let $\Delta(p, \varepsilon)$ be a small neighborhood of $p \in P$. Up to a change of coordinates, we may suppose $\varepsilon = 1$. Identify $\Delta(p, 1) \smallsetminus \{p\} = S^1 \times]0, 1]$ via polar coordinates (r, θ). Since the local model (1.20) is diagonal in the basis $\{e_k^j\}$, we see that the model Dirac operator on this disk

$$\emptyset_0^j = D^j - (D^j)^* : (\Omega^0 \oplus \Omega^2) \otimes E|_{\Delta(p,1)} \longrightarrow \Omega^1 \otimes E|_{\Delta(p,1)}$$

splits into the direct sum of its restrictions to the rank-one components generated by one of the $\{e_k^j\}$. Again, we have two cases: first, $k \in \{1, \dots r_j\}$ (regular case) and secondly $k \in \{r_j + 1, \dots r\}$ (singular case).

In the regular case, by definition the model Dirac operator on a rank-one component is just the operator

$$\emptyset = \mathrm{d} - \mathrm{d}^* : S^+ = \Omega^0 \oplus \Omega^2 \longrightarrow \Omega^1 = S^-,$$

which identifies to a projection of the real part of the usual Dirac operator on a product of two disks in $\mathbf{C}^2$ given by

$$\bar{\partial} - \bar{\partial}^* : \Omega^{0,0} \oplus \Omega^{0,2} \longrightarrow \Omega^{0,1}.$$

Since this is known to have an inverse for the Atiyah-Patodi-Singer boundary condition, the case of the regular part at a finite singularity follows.

On the singular component near a finite singularity, consider again the coordinate change $t = -\ln r \in \mathbf{R}^+$. The local model of D with respect to t is given by

$$D^j = d + i\bar{\mu}_k^j \mathrm{d}\theta + [\Re\mu_k^j - \beta_k^j]\frac{\mathrm{d}r}{r}$$

(see (1.20)). Notice that the rank of S^+ and that of S^- are both equal to 2: we trivialize them using the unit-norm sections $(1, r \, \mathrm{d}r \wedge \mathrm{d}\theta)$ and $(\mathrm{d}r, r\mathrm{d}\theta)$ respectively, so that both $S^+ \otimes E_{sing}$ and $S^- \otimes E_{sing}$ become isomorphic to $E_{sing} \oplus E_{sing}$ as Hermitian bundles. As we have seen in

Lemma 2.3, the space $H^1(\Delta(p,1), E_{sing})$ is equal to the model space of all sections ϕ having

$$\int_{\Delta(p,1)} \left(|\nabla\phi|^2 + \left| \frac{\phi}{r} \right|^2 \right) \, r\mathrm{d}r\mathrm{d}\theta < \infty.$$

By conformal invariance of the norm of 1-forms and $\mathrm{d}t = \mathrm{d}r/r$, this is

$$\int_{S^1 \times \mathbf{R}^+} (|\nabla\phi|^2 + |\phi|^2) \, \mathrm{d}t\mathrm{d}\theta < \infty,$$

with the norm of the 1-form $\nabla\phi$ measured with respect to the volume form $\mathrm{d}t\mathrm{d}\theta$. This latter is just the definition of the weighted Sobolev space $L_0^{2,1}(S^1 \times \mathbf{R}^+, E_{sing})$ with one derivative in L^2 and weight 0. In a similar way, the usual L^2-space of sections of E_{sing} on the disk is identified with the space $L_{-1}^2(S^1 \times \mathbf{R}^+, E_{sing})$ of L^2-sections with weight -1 on the half cylinder, for

$$\int_{\Delta(p,1)} |\phi|^2 \, r\mathrm{d}r\mathrm{d}\theta = \int_{S^1 \times \mathbf{R}^+} |\phi e^{-t}|^2 \mathrm{d}t\mathrm{d}\theta.$$

Hence in the trivialisation $(\mathrm{d}r, r\mathrm{d}\theta)$ of S^-, the usual L^2-space of 1-forms on the disk is identified with the weighted space

$$L_{-1}^2(S^1 \times \mathbf{R}^+, E_{sing} \oplus E_{sing}).$$

CLAIM 2.11. *Let (r, θ) be polar coordinates around $p = p^j$ as above. Let $k \in \{r_j + 1, \ldots, r\}$ and*

$$(f, g(r\,dr \wedge d\theta)) \otimes e_k^j \in C^\infty(\Delta \smallsetminus \{0\}, S^+ \otimes E_{sing}).$$

Then the value of the model Dirac operator $\displaystyle\not{\partial}^j$ on this section is

$$\left(\partial_r f + \frac{\Re\mu_k^j - \beta_k^j}{r} f - \frac{\partial_\theta + i\mu_k^j}{r} g \right) dr$$

$$+ \left(\frac{\partial_\theta + i\bar{\mu}_k^j}{r} f + \partial_r g - \frac{\Re\mu_k^j - \beta_k^j}{r} g \right) r\, d\theta.$$

In particular, in the unitary trivialisations $(1, r\ dr \wedge d\theta)$ and $(dr, r\,d\theta)$ of S^+ and S^-, the operator

$$r\not{\partial}^j = e^{-t}\not{\partial}^j$$

is translation-invariant with respect to the cylindrical coordinate t.

PROOF. This is a direct computation: for $f \otimes e_k^j$ it follows immediately from (1.20). For the image of $g(r\mathrm{d}r \wedge \mathrm{d}\theta) \otimes e_k^j$, consider first the

smooth form $\varphi dr \otimes e_k^j$ supported in a compact region of $\Delta \smallsetminus \{0\}$; then by the same formula we have

$$
\begin{aligned}
\langle \varphi dr \otimes e_k^j, (D^j)^* g(rdr \wedge d\theta) \otimes e_k^j \rangle &= \langle D^j(\varphi dr), g(rdr \wedge d\theta) \rangle \\
&= -\langle (\partial_\theta + i\bar{\mu}_k^j)\varphi dr \wedge d\theta, g(rdr \wedge d\theta) \rangle \\
&= -\frac{1}{r}\langle (\partial_\theta + i\bar{\mu}_k^j)\varphi, g \rangle \\
&= \frac{1}{r}\langle \varphi, (\partial_\theta + i\mu_k^j)g \rangle
\end{aligned}
$$

and thus the projection of $(D^j)^* g(rdr \wedge d\theta) \otimes e_k^j$ on the dr-component is $(\partial_\theta + i\mu_k^j)gdr \otimes e_k^j$. The other component is obtained taking a compactly supported smooth form $\psi rd\theta \otimes e_k^j$:

$$
\begin{aligned}
\langle \psi rd\theta \otimes e_k^j, (D^j)^* g(rdr \wedge d\theta) \otimes e_k^j \rangle &= \langle D^j(\psi rd\theta), g(rdr \wedge d\theta) \rangle \\
&= \left\langle \left(\partial_r + \frac{\Re\mu_k^j - \beta_k^j}{r} \right) \psi, g \right\rangle \\
&= \left\langle \psi, \left(-\partial_r + \frac{\Re\mu_k^j - \beta_k^j}{r} \right) g \right\rangle,
\end{aligned}
$$

and the formula of the claim follows. It implies that $r\slashed{\phi}^j$ is translation-invariant because $\partial_r = -\partial_t/r$. $\qquad\square$

By definition, the weight 0 is *critical* for $r\slashed{\phi}^j$ if and only if there exists a non-trivial solution of

$$
e^{-t}\slashed{\phi}^j(Ae^{-\nu t + in\theta}, Be^{-\nu t + in\theta}(r\ dr \wedge d\theta)) = 0
$$

with some constants $A, B \in \mathbf{C}$ and a constant $\nu \in \mathbf{C}$ such that $\Re\nu = 0$. Turning back to the coordinate r again, this is equivalent to having

$$
(2.18) \qquad r\slashed{\phi}^j(Ar^\nu e^{in\theta}, Br^\nu e^{in\theta}(r\ dr \wedge d\theta)) = 0.
$$

By [**17**], if 0 is not a critical weight, then the translation-invariant elliptic differential operator

$$
e^{-t}\slashed{\phi}^j : L_0^{2,1}(S^1 \times \mathbf{R}^+, S^+) \longrightarrow L_0^2(S^1 \times \mathbf{R}^+, S^-)
$$

is invertible, and thus so is

$$
\slashed{\phi}^j : L_0^{2,1}(S^1 \times \mathbf{R}^+, S^+) \longrightarrow L_{-1}^2(S^1 \times \mathbf{R}^+, S^-)
$$

since

$$
e^t : L_0^2 \longrightarrow L_{-1}^2
$$

is an isomorphism. Therefore, in order to establish the desired isomorphism in the singular case, we only need to check the weight 0 is not critical for $r\hat{\phi}^j$.

Applying the claim to the equation (2.18), we see that 0 is a critical weight if and only if the system of linear equations

$$(\nu + \Re\mu - \beta)A - i(n + \mu)B = 0$$
$$i(n + \bar{\mu})A + (\nu + \beta - \Re\mu)B = 0$$

has a non-trivial solution $(A, B) \in \mathbf{C}^2$ for some $\nu \in \mathbf{C}$ with $\Re\nu = 0$ (here we have omitted indices j and k of μ and β for simplicity). This system has a non-trivial solution if and only if the determinant formed by the coefficients is equal to 0:

$$\nu^2 - (\Re\mu - \beta)^2 - |n + \mu|^2 = 0.$$

Since $\Re\nu$ must be 0, this can only be the case if $\nu = \Re\mu - \beta = n + \mu = 0$. By assumption $0 \leq \beta < 1$, and n is an integer, therefore the only case this can hold is when $\beta = \mu = 0$, which is impossible, since we are looking at the singular component of the bundle. Therefore, there are no non-trivial solutions to (2.18), and 0 is not a critical weight.

2.2.2. Singularity at infinity. In this section the importance of the condition $\xi \notin \hat{P}$ will come out; therefore we write out the index ξ of our operators. A neighborhood of infinity in $\mathbf{C} \smallsetminus P$ is given by the complementary $\mathbf{C} \smallsetminus \Delta(R)$ of a large disk around 0. A natural choice of completion of this manifold is of course $\mathbf{C}$, with its standard metric $|dz|^2$. We choose to study the local model in the orthonormal basis $\{e_k^\infty\}$ defined in (1.21). This allows us to think of E as the trivial bundle $\mathbf{C}^r$ over $\mathbf{C} \smallsetminus \Delta(R)$, with standard hermitian metric on the fibers. By (1.30) this basis (up to a polynomial scaling factor) is a natural one for the Higgs-bundle point of view, so the deformation is that considered in (1.35), and the operator D_ξ near infinity is given (up to terms of order r^{-1}) by

$$D_\xi^\infty = \mathrm{d} + \frac{A - \xi\mathrm{Id}}{2}\mathrm{d}z + \frac{(A - \xi\mathrm{Id})^*}{2}\mathrm{d}\bar{z}$$

(see (1.23)), and a natural extension of it to all of $\mathbf{C}$ can be given by the same formula. This implies immediately that

$$\Phi_\xi^\infty = \frac{A - \xi\mathrm{Id}}{2}\mathrm{d}z + \frac{A^* - \bar{\xi}\mathrm{Id}}{2}\mathrm{d}\bar{z}$$

and $(D^\infty)^+ = \nabla$ (the trivial connection) on all of $\mathbf{C}$. For a section $\phi \in L^2(\Omega^0)$ supported in $\mathbf{C} \setminus \Delta(R)$, the condition $\Phi_\xi \phi \in L^2(\Omega^0)$ then automatically holds, and $(D^\infty_\xi)^+ \phi \in L^2$ is equivalent to $\nabla \phi \in L^2$. Therefore, on sections of Ω^0 supported on the complementary of $\Delta(R)$, the H^1-norm is equivalent to the usual Sobolev $L^{2,1}$-norm. A similar argument shows that for sections of Ω^2, the H^1-norm is also equivalent to the usual $L^{2,1}$-norm. Therefore, on all of $\mathbf{C}$, we must consider a natural extension of these functional spaces, namely $L^{2,1}(\mathbf{C}, \Omega^0 \oplus \Omega^2)$. In an analogous manner, on S^- we consider the extension $L^2(\mathbf{C}, \Omega^1)$ of $L^2(\mathbf{C} \setminus \Delta(R), \Omega^1)$. Therefore, we need to prove the

LEMMA 2.12. *On $\mathbf{C}$, the Dirac operator*

$$(2.19) \qquad \dirac^\infty_\xi = D^\infty_\xi - (D^\infty_\xi)^* : L^{2,1}(\Omega^0 \oplus \Omega^2) \longrightarrow L^2(\Omega^1)$$

is an isomorphism.

PROOF. Since A is supposed to be diagonal in this basis with eigenvalues ξ_l ($l = 1, \ldots, n'$), we may restrict ourselves to the study of the operator $D^\infty = \mathrm{d} + (\xi_l - \xi)/2\mathrm{d}z + (\bar\xi_l - \bar\xi)/2\mathrm{d}\bar z$. We need the following:

CLAIM 2.13. *Denote by Δ the plain Laplace operator $\nabla^*\nabla$ on forms. Then we have*

$$(2.20) \qquad \dirac^\infty_\xi (\dirac^\infty_\xi)^* = -\Delta - \frac{|\xi_l - \xi|^2}{4}.$$

PROOF. This is an easy computation. $\square$

Now recall that by the classical theory of the Laplace operator, $\Delta + \lambda^2$ with $\lambda > 0$ is an isomorphism

$$(2.21) \qquad L^{2,2}(\mathbf{C}, \Omega^j) \longrightarrow L^2(\mathbf{C}, \Omega^j).$$

This statement can be for example obtained passing to the Fourier transform $|\hat x|^2 + \lambda^2$ of this operator.

Coming back to our situation, the condition $\xi \notin \hat P$ means exactly that $\xi_l - \xi \neq 0$ for any $l = 1, \ldots, n'$. This immediately implies that (2.19) is surjective: indeed, clearly $Im((\dirac^\infty_\xi)^*) \subset L^{2,1}(\Omega^0 \oplus \Omega^2)$, and $\dirac^\infty_\xi (\dirac^\infty_\xi)^*$ is surjective by the isomorphism (2.21). For injectivity, note that a formula similar to (2.20) holds for the Laplace operator $(\dirac^\infty_\xi)^* \dirac^\infty_\xi$ as well. This in turn implies that the $L^{2,2}$-kernel of $\dirac^\infty_\xi$ vanishes. Elliptic regularity then shows that the $L^{2,1}$-kernel vanishes as well. $\square$

2.2.3. Compact perturbation. We wish to prove that near each one of the singularities the effect of passing from the global operator to its local model, i.e. subtracting the perturbation term only amounts to a compact operator $H^1(S^+ \otimes E) \to L^2(S^- \otimes E)$. This then finishes the proof of the Fredholm theorem, because the sum of a Fredholm operator and a finite number of compact operators is Fredholm.

Consider first the case of a singularity at a finite point. Recall from Lemma 2.3 that near p_j the space $H^1(S^+ \otimes E)$ is equal to the sum

$$L^{2,1}_{eucl}(S^+ \otimes E_{reg}) \oplus L^{2,1}_0(S^+ \otimes E_{sing}),$$

where $L^{2,1}_{eucl}$ is the usual Sobolev space on the disk of L^2-functions with one derivative in L^2 with respect to Euclidean metric, whereas $L^{2,1}_0$ is the weighted Sobolev space defined by

$$\int_{\Delta(p_j,\varepsilon)} \left(\left| \frac{\phi}{r} \right|^2 + |\nabla\phi|^2 \right) |dz|^2 \leq \infty.$$

Also, the order of growth of the 1-form perturbation term a^j with respect to Euclidean metric is by (2.11) at most $O(r^{-1+\delta})$, with $\delta > 0$. We need to prove that we have compact Sobolev multiplications for functions on the disk

(2.22) $$L^{2,1}_{eucl} \xrightarrow{a^j} L^2_{eucl}$$

and

(2.23) $$L^{2,1}_0 \xrightarrow{a^j} L^2_{eucl}.$$

Consider first (2.22): since the disk is a compact manifold, for any $2 < p < \infty$ the inclusion $L^{2,1}_{eucl} \hookrightarrow L^p_{eucl}$ is compact. On the other hand, $O(r^{-1+\delta})dr + O(r^{-1+\delta})rd\theta$ is in $L^{2+\varepsilon}_{eucl}$ for some $\varepsilon > 0$. Choose p such that $1/2 = 1/(2+\varepsilon) + 1/p$; (2.22) then follows immediately from the continuous multiplication $L^{2+\varepsilon}_{eucl} \times L^p_{eucl} \to L^2_{eucl}$. Now, we come to (2.23): this is an immediate consequence of the previous, for the weighted norm $L^{2,1}_0$ is stronger then $L^{2,1}_{eucl}$.

Next, let us treat the case of the singularity at infinity. In the coordinate $w = 1/z$ we have a second-order singularity on the disk $\Delta(0, 1/R)$. Let $w = \rho e^{i\vartheta}$; by (2.13) the perturbation is $O(\rho^{-1-\delta})$, and the H^1-norm of a function ϕ supported near infinity is given by

$$\int_{\mathbf{C}\smallsetminus\Delta(0,R)} (|\phi|^2 + |\nabla\phi|^2)\, |dz|^2 = \int_{\Delta(0,1/R)} \left(\left| \frac{\phi}{\rho^2} \right|^2 + |\nabla\phi|^2 \right) |dw|^2.$$

In particular, in the coordinate w this norm is also stronger then $L^{2,1}_{eucl}$, so we conclude from (2.22).

2.3. L^2-cohomology and Hodge theory

In this section we keep on supposing that we have on one side an integrable connection D with singularities in $P \cup \{\infty\}$, with prescribed behaviors at these points, given in regular singularities by (2.11) and at infinity by (2.13). In Theorem 2.6 we proved that the deformed operators $\rlap{\,/}\partial_\xi$ are Fredholm between the spaces H^1 and L^2; in particular their indices agree. We also showed that if the metric is harmonic then the kernel of the Dirac operator vanishes, hence the index of $\rlap{\,/}\partial_\xi$ is equal to the opposite of the dimension of the cokernel $Coker(\rlap{\,/}\partial_\xi)$, this operator being considered between functional spaces as in (2.14). This dimension is therefore a constant independent of ξ, and it follows from the implicit function theorem that the spaces $\hat{E}_\xi = Coker(\rlap{\,/}\partial_\xi)$ define a finite-rank smooth vector bundle $\hat{E}$ over $\hat{\mathbf{C}} \smallsetminus \hat{P}$, the rank being equal to the opposite of the index of (2.14). Here we wish to interpret this cokernel as the first cohomology of the elliptic complex

$$(2.24) \qquad L^2(\Omega^0 \otimes E) \xrightarrow{D_\xi} L^2(\Omega^1 \otimes E) \xrightarrow{D_\xi} L^2(\Omega^2 \otimes E),$$

(see Theorem 2.16), and also as the space of harmonic sections with respect to the Laplace operator of the adjoint Dirac operator $\rlap{\,/}\partial_\xi^*$ (Theorem 2.21).

Since the operators in (2.24) are unbounded, we need to define their domains. In this chapter C_0^∞ stands for smooth sections supported in a compact subset of $\mathbf{C} \smallsetminus P$.

DEFINITION 2.14. *The* maximal domain *of $D|_{\Omega^i}$ is*

$$Dom_{max}(D|_{\Omega^i}) = \{u \in L^2(\Omega^i) : Du \in L^2(\Omega^{i+1})\},$$

where $Du \in L^2$ is understood in the sense of currents, i.e. the functional $v \in C_0^\infty(\Omega^{i+1}) \mapsto \langle u, D^*v \rangle$ is continuous in the L^2-topology.

By local elliptic regularity, this amounts to the same thing as Du being an L^2-section. When it does not cause any confusion, we will simply write $Dom_{max}(\Omega^i)$ for $Dom_{max}(D|_{\Omega^i})$. It is easy to see that if we consider D on its maximal domain, then the kernel is a closed subspace of L^2, and the image of D on Ω^{i-1} is contained in the kernel

of D on Ω^i. The image of a general differential operator is however not always a closed subspace of the kernel.

DEFINITION 2.15. *For $i \in \{0, 1, 2\}$, the ith L^2-cohomology of D is $Ker(D|_{\Omega^i \otimes E})/Im(D|_{\Omega^{i-1} \otimes E})$, where both of these operators are considered with maximal domain, and the operators not shown in (2.24) are trivial. It is denoted by $L^2 H^1(D)$.*

Our aim is to obtain the following:

THEOREM 2.16. *The cokernel of $\partial\!\!\!/$ defined on $H^1(S^+ \otimes E)$ is equal to the first L^2-cohomology of D.*

PROOF. Recall that by definition

$$Coker(\partial\!\!\!/|_{H^1(S^+ \otimes E)}) = (Im(\partial\!\!\!/|_{H^1(S^+ \otimes E)}))^{\perp}$$

$$(2.25) \qquad = (Im(D|_{H^1(\Omega^0 \otimes E)}))^{\perp} \cap (Im(D^*|_{H^1(\Omega^2 \otimes E)}))^{\perp},$$

where $A^{\perp}$ stands for the L^2-orthogonal of the subspace $A \subset L^2$. Therefore, it is sufficient to prove the following lemmas:

LEMMA 2.17. *The maximal domain of*

$$D : L^2(\Omega^0 \otimes E) \longrightarrow L^2(\Omega^1 \otimes E)$$

is $H^1(\Omega^0 \otimes E)$. Similarly, the maximal domain of

$$D^* : L^2(\Omega^2 \otimes E) \longrightarrow L^2(\Omega^1 \otimes E)$$

is $H^1(\Omega^2 \otimes E)$. In particular, the maximal domain of

$$\partial\!\!\!/ : L^2(S^+ \otimes E) \longrightarrow L^2(S^- \otimes E)$$

is $H^1(S^+ \otimes E)$. Moreover, if this latter space is equipped with the norm $\|.\|_{H^1}$ defined in (2.1), then $\partial\!\!\!/$ is a bounded operator from $H^1(S^+ \otimes E)$ to $L^2(S^- \otimes E)$.

LEMMA 2.18. *We have*

$$(Im(D^*|_{H^1(\Omega^2 \otimes E)}))^{\perp} = Ker(D|_{Dom_{max^{(\Omega^1 \otimes E)}}}).$$

LEMMA 2.19. *The image of $D : H^1(\Omega^0 \otimes E) \to L^2(\Omega^1 \otimes E)$ is closed.*

Indeed, Lemmas 2.17 and 2.18 together with (2.25) imply that the cokernel is equal to

$$(Im(D|_{Dom_{max^{(\Omega^0 \otimes E)}}}))^{\perp} \cap Ker(D|_{Dom_{max^{(\Omega^1 \otimes E)}}}),$$

which in turn is identified to the first reduced L^2-cohomology of (2.24), i.e. to

$$Ker(D|_{\text{Dom}_{\max(\Omega^1 \otimes E)}})/\overline{Im(D|_{\text{Dom}_{\max(\Omega^0 \otimes E)}})},$$

where the bar over the image stands for the L^2-closure of that space. Lemma 2.19 now concludes the proof of Theorem 2.16.

PROOF. (Lemma 2.18) We first show the

CLAIM 2.20. *The adjoint of the unbounded operator*

(2.26) $$D^* : L^2(\Omega^2 \otimes E) \longrightarrow L^2(\Omega^1 \otimes E)$$

with domain $H^1(\Omega^2 \otimes E)$ *is the unbounded operator*

(2.27) $$D : L^2(\Omega^1 \otimes E) \longrightarrow L^2(\Omega^2 \otimes E)$$

with domain $Dom_{max}(\Omega^1 \otimes E)$.

PROOF. (Claim) It is clear that the formal adjoint of (2.26) is (2.27), we only need to prove its domain is $\text{Dom}_{\max}$. By definition, a section $u \in L^2(\Omega^1)$ is in the domain of the adjoint operator $\text{Dom}((D^*)^*)$ if and only if for all $v \in H^1(\Omega^2 \otimes E)$ we have

$$|\langle u, D^*v \rangle| \leq K \|v\|$$

with a constant K only depending on u. Now, since $v \in H^1$ and $u \in L^2$, by Claim 2.10 we can perform partial integration to the left-hand side of this formula. Therefore, u is in the domain of the adjoint operator if and only if the functional

$$v \mapsto \langle Du, v \rangle$$

is bounded in $L^2(\Omega^2 \otimes E)$. But this condition is equivalent to $Du \in L^2(\Omega^2 \otimes E)$, and the claim follows. $\qquad \square$

Lemma 2.18 now directly follows from the claim and the general fact that the cokernel of an unbounded operator is equal to the kernel of its adjoint. $\qquad \square$

PROOF. (Lemma 2.17) First we need to prove that for a section u of $L^2(\Omega^0 \otimes E)$ we have $Du \in L^2$ if and only if both $D^+u \in L^2$ and $\Phi u \in L^2$. The "if " direction being obvious, we concentrate ourselves on the opposite statement, and suppose in what follows that u is an L^2-function with $Du \in L^2$.

We first study the singularity at infinity. For $|u|$ sufficiently large, we have the point-wise estimate

$$|\Phi u| \leq 2K|u|,$$

where K is the maximal modulus of the eigenvalues of the matrix A. Therefore, $u \in L^2$ at infinity implies $\Phi u \in L^2$ at infinity, and consequently $D^+ u = Du - \Phi u \in L^2$ at infinity, and we are done.

Next, consider the case of a singularity at a finite point. In the orthonormal basis (1.17), the operators we study are equal, up to a perturbation term, to the local models (see (1.18), (1.19), (1.20))

$$(D^+)^j \phi = (\mathrm{d} + i\Re\mu_k^j \mathrm{d}\theta)\phi$$

$$\Phi^j \phi = [(\Re\mu_k^j - \beta_k^j)\frac{\mathrm{d}r}{r} + \Im\mu_k^j \mathrm{d}\theta]\phi$$

$$D^j \phi = [\mathrm{d} + i\bar{\mu}_k^j \mathrm{d}\theta + (\Re\mu_k^j - \beta_k^j)\frac{\mathrm{d}r}{r}]\phi$$

To simplify notation, from now on we drop the indices j and k. Note that because of Lemma 2.3, it is sufficient to prove that $\Phi^j \phi$ and $(D^+)^j \phi$ are in L^2. Notice also that since the perturbation a^j may mix the regular and singular components, a priori it is not sufficient to prove for example that $\phi_{reg} \in L^2$ and $D\phi_{reg} \in L^2$ imply $(D^+)^j \phi_{reg} \in L^2$, because $D\phi \in L^2$ does not imply directly $D\phi_{reg} \in L^2$ in the presence of a mixing perturbation term. However, remark that denoting by $a_{r,r}^j$ the part of the endomorphism a^j that takes the regular component into the regular one, and $a_{r,s}^j, a_{s,r}^j, a_{s,s}^j$ the other parts, we have

(2.28)
$$\int_{\Delta(p_j,\varepsilon)} |(D^j + a^j)\phi|^2 = \int_{\Delta(p_j,\varepsilon)} |(D^j + a_{r,r}^j)\phi_{reg} + a_{s,r}^j\phi_{sing}|^2$$

$$+ \int_{\Delta(p_j,\varepsilon)} |(D^j + a_{s,s}^j)\phi_{sing} + a_{r,s}^j\phi_{reg}|^2$$

$$\geq \int_{\Delta(p_j,\varepsilon)} |D^j\phi_{reg}|^2 + |D^j\phi_{sing}|^2$$

$$- |a^j\phi_{reg}|^2 - |a^j\phi_{sing}|^2,$$

and this estimate shows that we can treat the two components separately: the left-hand side is finite by hypothesis, whereas the integrals of $|a^j\phi_{reg}|^2$ and $|a^j\phi_{sing}|^2$ by Kato's inequality and (2.12); hence the same thing holds for the integrals of $|D^j\phi_{reg}|^2$ and $|D^j\phi_{reg}|^2$.

On the regular component, the above expressions simplify to $D^j = (D^+)^j = \nabla$ (the trivial connection), and $\Phi^j = 0$. What we need to

show is that $\phi_{reg}, D\phi_{reg} \in L^2$ implies $\nabla\phi_{reg} \in L^2$, if $D = \nabla + a^j$ with $a^j = O(r^{-1+\delta})$. Recall that by Kato's inequality and (2.12) with $\varepsilon > 0$ chosen sufficiently small we have

$$\int_{\Delta(p_j,\varepsilon)} |a^j\phi_{reg}|^2 \leq \int_{\Delta(p_j,\varepsilon)} |D\phi_{reg}|^2 + \int_{\Delta(p_j,\varepsilon)\smallsetminus\Delta(p_j,\varepsilon/2)} |\phi_{reg}|^2.$$

It follows that

$$\int_{\Delta(p_j,\varepsilon)} |\nabla\phi_{reg}|^2 \leq \int_{\Delta(p_j,\varepsilon)} |D\phi_{reg}|^2 + \int_{\Delta(p_j,\varepsilon)} |a^j\phi_{reg}|^2$$
$$< 2\int_{\Delta(p_j,\varepsilon)} |D\phi_{reg}|^2 + 2\int_{\Delta(p_j,\varepsilon)} |\phi_{reg}|^2.$$

Now by the hypothesis $\phi, D\phi \in L^2$, the right-hand side is finite. Therefore $\nabla\phi \in L^2$ as we wished to show.

Consider now the singular case: again, we need to show that if we have a section $\phi \in L^2$ such that $D\phi \in L^2$, then $D^+\phi_{sing}, \Phi\phi_{sing} \in L^2$. Here, usual elliptic regularity does not give the claim, because we need to deduce that $\phi_{sing}/r \in L^2$. From now on, we write $\phi = \phi_{sing}$ to lighten notation. Decompose ϕ into its Fourier-series near p^j:

$$\phi(r,\theta) = \sum_{n=-\infty}^{\infty} \phi_n(r)e^{in\theta}$$

Choosing ε sufficiently small, we can make the perturbation term a^j be smaller on $\Delta(p_j,\varepsilon)$ then ν/r for any $\nu > 0$. Write first the $d\theta$-term of $D^j\phi$:

$$D_\theta^j\phi = (\partial_\theta + i\bar\mu)\phi d\theta = id\theta \sum_{n=-\infty}^{\infty} (n+\bar\mu)\phi_n(r)e^{in\theta}.$$

By this and the estimate on the perturbation, we infer that

$$\|(D_\theta^j + a^j)\phi\|_{L^2(\Delta(p_j,\varepsilon))}^2 \geq \|D_\theta^j\phi\|_{L^2(\Delta(p_j,\varepsilon))}^2 - \|\nu\phi/r\|_{L^2(\Delta(p_j,\varepsilon))}^2$$

(2.29)
$$= \int_{\Delta(p,\varepsilon)} \sum_{n=-\infty}^{\infty} (|n+\bar\mu|^2 - \nu^2)\frac{|\phi_n(r)|^2}{r^2}$$

$$= \int_{\Delta(p,\varepsilon)} \sum_{n=-\infty}^{\infty} (|n+\Re\mu|^2 - \nu^2 + |\Im\mu|^2)\frac{|\phi_n(r)|^2}{r^2}.$$

By Hypothesis 1.28 we have $\Re\mu \notin \mathbf{Z}$, and so if ν is sufficiently small, then the last expression can be bounded from below by

$$(2.30) \qquad \frac{1}{2}\int_{\Delta(p,\varepsilon)}\sum_{n=-\infty}^{\infty}(|n+\Re\mu|^2+|\Im\mu|^2)\frac{|\phi_n(r)|^2}{r^2}$$

$$=\frac{1}{2}\int_{\Delta(p,\varepsilon)}|(D_\theta^+)^j\phi|^2+|\Phi_\theta^j\phi|^2.$$

As in the regular case, by (2.12) the left-hand side of (2.29) is finite, so we see that $(D_\theta^+)^j\phi \in L^2$ and $\Phi_\theta^j\phi \in L^2$. The d$r$-part $\Phi_r^j\phi$ of $\Phi^j\phi$ is in L^2 if and only if

$$\int_{\Delta(p,\varepsilon)}|\Re\mu-\beta|^2\frac{|\phi(r)|^2}{r^2}<\infty.$$

Again by our main hypothesis $\Re\mu \notin \mathbf{Z}$ there exists a constant $K > 0$ such that

$$\sum_{n=-\infty}^{\infty}|\Re\mu-\beta|^2\frac{|\phi_n(r)|^2}{r^2}\leq K\sum_{n=-\infty}^{\infty}|n+\Re\mu|^2\frac{|\phi_n(r)|^2}{r^2}.$$

As we have already seen, this last expression is integrable, therefore $\Phi^j\phi \in L^2$. Since the perturbation is negligible compared to the behavior $O(r^{-1})$ of (2.30), we then also have $\Phi\phi \in L^2$. We conclude using $D^+\phi = D\phi - \Phi\phi$.

By duality, the case of a 2-form $v\mathrm{d}z \wedge \mathrm{d}\bar{z}$ is settled the same way. The general case (that of $S^+ \otimes E$) then follows from Lemma 2.9. The fact that

$$\displaystyle{\not{D}}: H^1(S^+ \otimes E) \longrightarrow L^2(S^- \otimes E)$$

is bounded, is then immediate (and has already been pointed out, see (2.14)). $\qquad\square$

PROOF. (Lemma 2.19) This is immediate from Theorem 1 and Claim 2.9. $\qquad\square$

We have established lemmata 2.18, 2.19 and 2.17, hence we finished the proof of Theorem 2.16. $\qquad\square$

THEOREM 2.21. *The first L^2-cohomology of the complex (2.24) is canonically isomorphic to the kernel of the adjoint Dirac operator*

$$(2.31) \qquad\qquad \not{D}_\xi^*: L^2(S^- \otimes E) \longrightarrow L^2(S^+ \otimes E)$$

on its domain, or alternatively to the kernel of the Laplace operator

$$(2.32) \quad \Delta_\xi = \not{D}_\xi\not{D}_\xi^* = -D_\xi D_\xi^* - D_\xi^* D_\xi : L^2(S^- \otimes E) \longrightarrow L^2(S^- \otimes E)$$

on its domain.

PROOF. By duality, we get from Lemma 2.18 that

$$(Im(D|_{H^1(\Omega^0 \otimes E)}))^\perp = ker(D^*|_{\text{Dom}_{\max}(\Omega^1 \otimes E)}),$$

and this implies

$$coKer(\not{\partial}|H^1(S^+ \otimes E)) = ker(D^*|_{\text{Dom}_{\max}(\Omega^1 \otimes E)}) \cap ker(D|_{\text{Dom}_{\max}(\Omega^1 \otimes E)})$$
$$= ker(\not{\partial}^*|_{\text{Dom}_{\max}(\Omega^1 \otimes E)}).$$

It remains to show that this latter is equal to $ker(\not{\partial}\not{\partial}^*|_{\text{Dom}_{\max}(\Omega^1 \otimes E)})$. It is clear that

$$ker(\not{\partial}\not{\partial}^*|_{\text{Dom}_{\max}(\Omega^1 \otimes E)}) \supseteq ker(\not{\partial}^*|_{\text{Dom}_{\max}(\Omega^1 \otimes E)}).$$

Suppose now $u \in L^2(\Omega^1 \otimes E)$ satisfies $\not{\partial}\not{\partial}^* u = 0$. This means that

$$\not{\partial}^* u \in Ker(\not{\partial}) \subset \text{Dom}_{\max}(\not{\partial}) = H^1(S^+ \otimes E)$$

by Lemma 2.17. Vanishing of the L^2-kernel of $\not{\partial}$ on $H^1(S^+ \otimes E)$ (c.f. Theorem 2.6) gives $\not{\partial}^* u = 0$, that is $u \in Ker(\not{\partial}^*)$, whence

$$ker(\not{\partial}\not{\partial}^*|_{\text{Dom}_{\max}(\Omega^1 \otimes E)}) \subseteq ker(\not{\partial}^*|_{\text{Dom}_{\max}(\Omega^1 \otimes E)}).$$

$\square$

Finally, let us introduce the norm

$$\|f\|_{H^2(S^+ \otimes E)} = \int_{\mathbf{C}} |f|^2 + |(\nabla^+)^* \nabla^+ f|^2 + |(\Phi \otimes)^* \Phi \otimes f|^2$$

and the corresponding function space

$$H^2(S^+ \otimes E) = \{f : \quad \|f\|_{H^2(S^+ \otimes E)} < \infty\}$$

Then we have the following.

THEOREM 2.22. *The domain of the Laplace operator* $\Delta_\xi = \not{\partial}_\xi^* \not{\partial}_\xi$ *is* $H^2(S^+ \otimes E)$. *It defines a Hilbert-space isomorphism*

$$H^2(S^+ \otimes E) \longrightarrow L^2(S^+ \otimes E).$$

PROOF. The fact that Δ_ξ is a well-defined bounded operator on $H^2(S^+ \otimes E)$ follows from the Weitzenböck formula (2.16). Its maximal domain is the set of $u \in L^2(S^+ \otimes E)$ such that $\not{\partial}_\xi u \in \text{Dom}_{\max}(\not{\partial}_\xi^*)$. This latter is, by computations similar to Lemma 2.17, the Sobolev space $H^1(S^- \otimes E)$ is with 1 derivative in L^2, and weight -1 on the irregular component near logarithmic singularities like in Corollary 2.5.

We deduce that the maximal domain of Δ_ξ is $H^2(S^+ \otimes E)$, and that it splits as

$$H^2(S^+ \otimes E) \xrightarrow{\slashed{\partial}_\xi} H^1(S^- \otimes E) \xrightarrow{\slashed{\partial}_\xi^*} L^2(S^+ \otimes E).$$

Exactly as in Theorem 2.6, the first map is Fredholm with vanishing kernel from the Sobolev space $H^2(S^+ \otimes E)$ into $H^1(S^- \otimes E)$, both space being endowed with the L^2-inner product. This with the identity $Im(\slashed{\partial}_\xi)^\perp = Ker(\slashed{\partial}_\xi^*)$ implies that $Ker(\Delta_\xi) = \{0\}$ and that $Im(\Delta_\xi) = Im(\slashed{\partial}_\xi^*) = Ker(\slashed{\partial}_\xi)^\perp = L^2(S^+ \otimes E)$. Therefore, Δ_ξ is a bounded bijective operator from $H^2(S^+ \otimes E)$ to $L^2(S^+ \otimes E)$. By the closed graph theorem, we conclude that its inverse is also bounded. □

2.4. Properties of the Green's operator

DEFINITION 2.23. *Let us call the bounded linear inverse of $\slashed{\partial}_\xi^* \slashed{\partial}_\xi$ provided by Theorem 2.22 the* Green's operator *of the Dirac-Laplace operator, and denote it by*

$$G_\xi : L^2(S^+ \otimes E) \longrightarrow H^2(S^+ \otimes E).$$

In this section we list the properties of this operator that we will need in later chapters.

LEMMA 2.24. *G_ξ is diagonal with respect to the decomposition $S^+ \otimes E = (\Omega^0 \otimes E) \oplus (\Omega^2 \otimes E)$.*

PROOF. Since G_ξ is the inverse of Δ_ξ, it is sufficient to prove the statement for this latter operator. This comes from the identity

$$\Delta_\xi = \slashed{\partial}_\xi^* \slashed{\partial}_\xi = (D_\xi^* - D_\xi)(D_\xi - D_\xi^*) = -D_\xi^* D_\xi - D_\xi D_\xi^*,$$

which is satisfied since D_ξ is flat. □

LEMMA 2.25. *There exist $K, K' > 0$ such that for $|\xi|$ sufficiently large and for any positive spinor $\psi \in H^1(S^+ \otimes E)$, the following estimates hold:*

(2.33) $$\|G_\xi \psi\|_{L^2(\mathbf{C})} \leq K |\xi|^{-2} \|\psi\|_{L^2(\mathbf{C})}$$

(2.34) $$\|G_\xi \psi\|_{H^1(\mathbf{C})} \leq K' |\xi|^{-1} \|\psi\|_{L^2(\mathbf{C})}$$

PROOF. Since by definition, for any ψ the positive spinor $G_\xi \psi$ is the solution φ of

$$\Delta_\xi \varphi = \psi,$$

the estimates (2.33) and (2.34) can be rewritten respectively as

$$(2.35) \qquad\qquad \|\varphi\|_{L^2(\mathbf{C})} \leq K|\xi|^{-2}\|\Delta_\xi\varphi\|_{L^2(\mathbf{C})}$$

$$(2.36) \qquad\qquad \|\varphi\|_{H^1(\mathbf{C})} \leq K'|\xi|^{-1}\|\Delta_\xi\varphi\|_{L^2(\mathbf{C})}.$$

Call ξ-*energy* of φ over all $\mathbf{C}$ the quantity

$$(2.37) \qquad\qquad E(\xi;\varphi) = \int_{\mathbf{C}} |\nabla_\xi^+\varphi|^2 + |\Phi_\xi \otimes \varphi|^2 |\mathrm{d}z|^2.$$

By partial integration, the Weitzenböck formula (2.16) and Cauchy's inequality we have

$$(2.38) \qquad\qquad E(\xi;\varphi) = \int_{\mathbf{C}} \langle \varphi, \Delta_\xi\varphi \rangle |\mathrm{d}z|^2$$
$$\leq \|\varphi\|_{L^2}\|\Delta_\xi\varphi\|_{L^2}.$$

Now, as we will see from (4.46), on the complementary of a finite union of disks $\Delta(q_k(\xi), \varepsilon_0|\xi|^{-1})$ we have the point-wise lower bound

$$(2.39) \qquad\qquad |\Phi_\xi \otimes \varphi|^2 \geq c|\xi|^2|\varphi|^2$$

for some $c > 0$. Furthermore, we can choose ε_0 sufficiently small so that the balls $\Delta(q(\xi), 2\varepsilon_0|\xi|^{-1})$ are disjoint and do not meet P for $|\xi|$ large. Setting

$$B_\xi := \bigcup_{q(\xi)\in\Sigma_\xi} \Delta(q(\xi), \varepsilon_0|\xi|^{-1})$$

we then deduce the estimation

$$(2.40) \qquad \int_{\mathbf{C}\smallsetminus B_\xi} |\Phi_\xi \otimes \varphi|^2 |\mathrm{d}z|^2 \geq c|\xi|^2 \int_{\mathbf{C}\smallsetminus B_\xi} |\varphi|^2 |\mathrm{d}z|^2.$$

Of course, extending this inequality over the disks $\Delta(q(\xi), \varepsilon_0|\xi|^{-1})$ is not possible, since Φ_ξ has a zero in $q(\xi)$. However, the integral of $|\Phi_\xi \otimes \varphi|^2 + |\nabla_\xi^+\varphi|^2$ does control $|\xi|^2$ times that of $|\varphi|^2$ on the whole plane; that is, we have:

CLAIM 2.26. *There exists $c > 0$ such that for $|\xi|$ sufficiently large and for any spinor φ we have*

$$(2.41) \qquad\qquad E(\xi;\varphi) \geq c|\xi|^2 \int_{\mathbf{C}} |\varphi|^2 |dz|^2$$

PROOF. By Kato's inequality $E(\xi;\varphi)$ can be bounded from below by

$$\int_{\mathbf{C}} |\Phi_\xi \otimes \varphi|^2 + |\mathrm{d}|\varphi||^2 |\mathrm{d}z|^2.$$

By (2.40), it only remains to show that for any $q(\xi) \in \Sigma_\xi$ this integral bounds from above $c|\xi|^2 \int_{\Delta(q(\xi), \varepsilon_0|\xi|^{-1})} |\varphi|^2 |dz|^2$, for some $c > 0$ (not necessarily the same as before). But since on the annulus

$$\Delta(q(\xi), 2\varepsilon_0|\xi|^{-1}) \smallsetminus \Delta(q(\xi), \varepsilon_0|\xi|^{-1})$$

we already have the estimation (2.39), this is just a consequence of (2.12) applied at the point $q(\xi)$ instead of p_j to the function $g = |\varphi|$, with $\varepsilon = \varepsilon_0|\xi|^{-1}$ and $\delta = 0$. $\qquad\square$

By the claim and (2.38), we have

$$c|\xi|^2\|\varphi\|^2_{L^2(\mathbf{C})} \le \|\varphi\|_{L^2(\mathbf{C})}\|\Delta_\xi\varphi\|_{L^2(\mathbf{C})},$$

and after dividing both sides by $\|\varphi\|_{L^2(\mathbf{C})}$, we get (2.35).

Plugging (2.35) into (2.38), we obtain

(2.42) $$E(\xi;\varphi) \le K|\xi|^{-2}\|\Delta_\xi\varphi\|^2_{L^2(\mathbf{C})}.$$

On the other hand, by the definitions

$$\nabla_\xi^+ = \nabla^+ - \frac{\xi}{2}dz + \frac{\bar{\xi}}{2}d\bar{z}$$

$$\Phi_\xi = \Phi - \frac{\xi}{2}dz - \frac{\bar{\xi}}{2}d\bar{z}$$

we obtain the point-wise bounds

$$\frac{1}{2}|\Phi \otimes \varphi|^2 - \frac{3}{2}|\xi|^2|\varphi|^2 \le |\Phi_\xi \otimes \varphi|^2 \le 2|\Phi \otimes \varphi|^2 + |\xi|^2|\varphi|^2$$

$$\frac{1}{2}|\nabla^+\varphi|^2 - \frac{3}{2}|\xi|^2|\varphi|^2 \le \left|\nabla_\xi^+ \otimes \varphi\right|^2 \le 2|\nabla^+\varphi|^2 + |\xi|^2|\varphi|^2$$

and therefore

(2.43)
$$\frac{1}{2}\|\varphi\|^2_{H^1(\mathbf{C})} - (3|\xi|^2+1)\|\varphi\|^2_{L^2(\mathbf{C})} \le E(\xi;\varphi) \le 2\|\varphi\|^2_{H^1(\mathbf{C})} + (2|\xi|^2+1)\|\varphi\|^2_{L^2(\mathbf{C})}.$$

Putting together this with (2.42) and (2.35), we get

$$\|\varphi\|^2_{H^1(\mathbf{C})} \le 2E(\xi;\varphi) + (6|\xi|^2 + 2)\|\varphi\|^2_{L^2(\mathbf{C})}$$
$$\le 2E(\xi;\varphi) + 7|\xi|^2\|\varphi\|^2_{L^2(\mathbf{C})}$$
$$\le (2K + 7K^2)|\xi|^{-2}\|\Delta_\xi\varphi\|^2_{L^2(\mathbf{C})},$$

whence (2.36). $\qquad\square$

We now investigate what happens to the Green's operator when ξ is close to one of the points of $\hat{P}$.

LEMMA 2.27. *There exist $K, K' > 0$ such that for $|\xi - \xi_l|$ sufficiently small and for any positive spinor $\psi \in H^1(S^+ \otimes E)$, the following estimates hold:*

$$(2.44) \qquad \|G_\xi \psi\|_{L^2(\mathbf{C})} \leq K|\xi - \xi_l|^{-2} \|\psi\|_{L^2(\mathbf{C})}$$

$$(2.45) \qquad \|\not{\partial}_\xi G_\xi \psi\|_{L^2(\mathbf{C})} \leq K''|\xi - \xi_l|^{-1} \|\psi\|_{L^2(\mathbf{C})}$$

PROOF. Analogous to Lemma 2.25. Notice that by partial integration and the Weitzenböck formula (2.16) one has

$$\|\not{\partial}_\xi \varphi\|_{L^2(\mathbf{C})}^2 = E(\xi; \varphi)$$

for any positive spinor φ. Using this and setting $G_\xi \psi = \varphi$ the inequalities to prove can be rewritten as

$$(2.46) \qquad \|\varphi\|_{L^2(\mathbf{C})} \leq K|\xi - \xi_l|^{-2} \|\Delta_\xi \varphi\|_{L^2(\mathbf{C})}$$

$$(2.47) \qquad E(\xi; \varphi) \leq K''|\xi - \xi_l|^{-2} \|\Delta_\xi \varphi\|_{L^2(\mathbf{C})}^2.$$

The behavior (4.62) of the Higgs field shows that outside of a finite union of disks $\Delta(q_k(\xi), \varepsilon_0|\xi - \xi_l|^{-1})$ there exists $c > 0$ for which we have the point-wise lower bound

$$(2.48) \qquad |\Phi_\xi \otimes \varphi|^2 \geq c|\xi - \xi_l|^2 |\varphi|^2.$$

It follows that denoting by B_ξ the union of all the above mentioned disks where this estimate may fail, we have the inequality

$$(2.49) \qquad \int_{\mathbf{C} \smallsetminus B_\xi} |\Phi_\xi \otimes \varphi|^2 |\mathrm{d}z|^2 \geq c|\xi - \xi_l|^2 \int_{\mathbf{C} \smallsetminus B_\xi} |\varphi|^2 |\mathrm{d}z|^2.$$

It is not possible to extend this inequality to the whole plane; however, we have again

CLAIM 2.28. *There exists $c > 0$ such that for $|\xi - \xi_l|$ sufficiently small and for any spinor φ we have*

$$(2.50) \qquad E(\xi; \varphi) \geq c|\xi - \xi_l|^2 \int_{\mathbf{C}} |\varphi|^2 |dz|^2$$

PROOF. Similar to Claim 2.26, using Kato's inequality and (2.12) rescaled conveniently by the homothety $w = (\xi - \xi_l)z$. $\qquad \square$

This together with (2.38) then shows

$$c|\xi - \xi_l|^2 \|\varphi\|_{L^2(\mathbf{C})}^2 \leq \|\varphi\|_{L^2(\mathbf{C})} \|\Delta_\xi \varphi\|_{L^2(\mathbf{C})},$$

which gives us (2.46). Plugging this back into (2.38), we obtain (2.47).

$\square$

2.5. Exponential decay results for harmonic spinors

In this section we give some analytic properties of Δ_ξ-harmonic spinors. They will be needed in Section 3.1, where we study the transformed flat connection. More precisely, they will allow us to multiply any L^2 harmonic section by exponential factor so that the result remains in L^2. They will also be of use in the computation of the parabolic weights of the transform in Section 4.6.

First we set some further notation. Fix $\xi \in \hat{\mathbf{C}} \smallsetminus \hat{P}$, and let φ be a harmonic negative spinor with respect to $\bar{\partial}_\xi \bar{\partial}_\xi^*$ and $p \in \mathbf{C} \smallsetminus P$ any point of the plane. Finally, for any spinor ψ (not necessarily harmonic), call ξ-energy of ψ in the disk $\Delta(p, \varepsilon)$ the quantity

$$(2.51) \qquad E(p, \varepsilon, \xi; \psi) = \int_{\Delta(p,\varepsilon)} |\nabla_\xi^+ \psi|^2 + |\Phi_\xi \otimes \psi|^2.$$

LEMMA 2.29. *Suppose that there exists $\varepsilon_0 > 0$, $R > 0$ and $c > 0$ such that the disk $\Delta(p, (R+1)\varepsilon_0)$ is disjoint from P, and all of the eigenvalues of θ_ξ in any point of this disk are bounded below in absolute value by $c > 0$. Under these assumptions, we have the inequality*

$$(2.52) \qquad E(p, \varepsilon_0, \xi; \varphi) \le e^{-2cR\varepsilon_0} \left(2\|\varphi\|_{H^1(\mathbf{C})}^2 + (2|\xi|^2 + 1)\|\varphi\|_{L^2(\mathbf{C})}^2 \right).$$

PROOF. Denote by $C(p, r)$ the boundary of $\Delta(p, r)$, and by $\frac{\partial}{\partial n}$ an outward-pointing unit normal vector to it. Stokes' formula gives

$$E(p, r, \xi; \varphi) = \int_{\Delta(p,r)} \left((\nabla_\xi^+)^* \nabla_\xi^+ \varphi + (\Phi_\xi \otimes)^* \Phi_\xi \otimes \varphi, \varphi \right)$$

$$+ \int_{C(p,r)} \left((\nabla_\xi^+)_{\frac{\partial}{\partial n}} \varphi, \varphi \right) r \mathrm{d}\theta.$$

Since φ is Δ_ξ-harmonic, the Weitzenböck formula (2.16) implies that the first term on the right-hand side vanishes. Therefore, by the tic-tac-toe inequality, we have

$$E(p, r, \xi; \varphi) \le \frac{1}{2} \int_{C(p,r)} \frac{1}{c} |\nabla^+ \varphi|^2 + c|\varphi|^2 r \mathrm{d}\theta.$$

On the other hand, we have

$$\frac{\mathrm{d}E(p, r, \xi; \varphi)}{\mathrm{d}r} = \int_{C(p,r)} \left| \nabla_\xi^+ \varphi \right|^2 + |\Phi_\xi \otimes \varphi|^2 r \mathrm{d}\theta.$$

By assumption, for $r \le (R+1)\varepsilon_0$ we have the estimate

$$\int_{C(p,r)} |\Phi_\xi \otimes \varphi|^2 r \mathrm{d}\theta \ge c^2 \int_{C(p,r)} |\varphi|^2 r \mathrm{d}\theta.$$

Putting together these estimates, we see that

$$\frac{\mathrm{d}E(p, r, \xi; \varphi)}{\mathrm{d}r} \geq 2cE(p, r, \xi; \varphi),$$

whence

$$\frac{\mathrm{d}\log E(p, r, \xi; \varphi)}{\mathrm{d}r} \geq 2c.$$

Integrating this inequality from $r = \varepsilon_0$ to $r = (R + 1)\varepsilon_0$, we obtain

$$\log E(p, \varepsilon_0, \xi; \varphi) \leq 2c[\varepsilon_0 - (R + 1)\varepsilon_0] + \log E(p, (R + 1)\varepsilon_0, \xi; \varphi).$$

Taking exponential of both sides, we get

$$E(p, \varepsilon_0, \xi; \varphi) \leq e^{-2cR\varepsilon_0} E(p, (R + 1)\varepsilon_0, \xi; \varphi)$$
$$\leq e^{-2cR\varepsilon_0} E(\xi; \varphi),$$

and we conclude using (2.43).   □

Next, we use the above lemma to obtain exponential decay results in terms of ξ for the energy of harmonic spinors when ξ is large, first in a fixed disk of $\mathbf{C}$ away from the singularities P, then near infinity in $\mathbf{C}$. In the first case, the statement is as follows.

LEMMA 2.30. *Let $p \in \mathbf{C} \setminus P$ be arbitrary, and let $\varepsilon_0 > 0$ be such that the distance between p and P is at least $3\varepsilon_0$. Then for $|\xi|$ sufficiently large we have the estimate*

$$\|\varphi\|^2_{H^1(\Delta(p, \varepsilon_0))} \leq e^{-\varepsilon_0|\xi|/3} \|\varphi\|^2_{H^1(\mathbf{C})}$$

for any Δ_ξ-harmonic spinor φ.

PROOF. Since p is away from P, in the Higgs field $\theta_\xi = \theta - \xi\mathrm{d}z/2$ the term θ is bounded on $\Delta(p, 2\varepsilon_0)$. Therefore, if $|\xi|$ is sufficiently large, then the eigenvalues of θ_ξ on this disk are bounded below in absolute value by $|\xi|/4$. Apply Lemma 2.29 with $R = 1$ and $c = |\xi|/4$ to get

$$E(p, \varepsilon_0, \xi; \varphi) \leq e^{-\varepsilon_0|\xi|/2} \left(2\|\varphi\|^2_{H^1(\mathbf{C})} + (2|\xi|^2 + 1)\|\varphi\|^2_{L^2(\mathbf{C})}\right)$$
$$\leq 5e^{-\varepsilon_0|\xi|/2}|\xi|^2\|\varphi\|^2_{H^1(\mathbf{C})}$$
$$\leq \frac{1}{33} e^{-\varepsilon_0|\xi|/3}\|\varphi\|^2_{H^1(\mathbf{C})}$$

for ξ sufficiently large. On the other hand, we have

$$\|\varphi\|^2_{H^1(\Delta(p,\varepsilon_0))} = \int_{\Delta(p,\varepsilon_0)} |\varphi|^2 + |\nabla^+\varphi|^2 + |\Phi \otimes \varphi|^2$$

(2.53)
$$\leq \int_{\Delta(p,\varepsilon_0)} 2|\xi|^2|\varphi|^2 + \left|\nabla^+_\xi \varphi\right|^2 + |\Phi_\xi \otimes \varphi|^2$$

$$\leq 33\, E(p, \varepsilon_0, \xi; \varphi),$$

where the last line is a consequence of $|\Phi_\xi \otimes \varphi|^2 \geq |\xi|^2|\varphi|^2/16$ in $\Delta(p, \varepsilon_0)$. Putting together these two estimates, we get the lemma. $\qquad\square$

In the second case, we have the following statement.

LEMMA 2.31. *For any $\xi \notin \hat{P}$ there exists $R_0 = R_0(\xi) > 0$, $K = K(\xi) > 0$ and $c = c(\xi) > 0$ such that for any Δ_ξ-harmonic spinor φ and all $R > R_0$ the following estimate holds:*

$$\|\varphi\|^2_{H^1(\mathbf{C}\smallsetminus\Delta(0,2R))} \leq Ke^{-Rc}\|\varphi\|^2_{H^1(\mathbf{C})}.$$

Furthermore, if $|\xi|$ is sufficiently large, we can choose $c = |\xi|/3$ and R_0, K constants independent of ξ.

PROOF. The proof is an amalgam of that of Lemmata 2.29 and 2.30. Define the *ξ-energy at infinity* of a spinor by the integral

(2.54)
$$E(\infty, R, \xi; \varphi) = \int_{\mathbf{C}\smallsetminus\Delta(0,R)} |\nabla^+_\xi\varphi|^2 + |\Phi_\xi \otimes \varphi|^2.$$

Choose $R_0 > 0$ and c_0 such that for $|z| > R_0$ the eigenvalues of $\theta_\xi(z)$ are all bigger in absolute value then c_0. Clearly, such a choice is possible because $\xi \notin \hat{P}$. Moreover, for $|\xi|$ sufficiently large one can put $c_0 = |\xi|/4$ and R_0 a constant only depending on the initial data θ. For $r \geq R_0$, we have the estimate

$$-E(\infty, r, \xi; \varphi) \geq -\frac{1}{2}\int_{C(0,r)} \frac{1}{c_0}\left|\nabla^+_\xi\varphi\right|^2 + c_0|\varphi|^2 r\mathrm{d}\theta.$$

On the other hand, we have

$$\frac{\mathrm{d}E(\infty, r, \xi; \varphi)}{\mathrm{d}r} = -\int_{C(0,r)} \left|\nabla^+_\xi\varphi\right|^2 + |\Phi_\xi \otimes \varphi|^2 r\mathrm{d}\theta.$$

By assumption, we have also

$$\int_{C(0,r)} |\Phi_\xi \otimes \varphi|^2 r\mathrm{d}\theta \geq c_0^2\int_{C(0,r)} |\varphi|^2 r\mathrm{d}\theta.$$

Putting together these estimates, we see that for $r \geq R_0$

$$\frac{\mathrm{d}E(\infty, r, \xi; \varphi)}{\mathrm{d}r} \leq -2c_0 E(\infty, r, \xi; \varphi),$$

whence
$$\frac{\mathrm{d}\log E(\infty, r, \xi; \varphi)}{\mathrm{d}r} \leq -2c_0.$$
Integrating this inequality from R to $2R$ and using (2.43), we obtain
$$\begin{aligned}
E(\infty, 2R, \xi; \varphi) &\leq E(\xi; \varphi)e^{-Rc_0} \\
&\leq (|\xi|^2 + 3)e^{-Rc_0}\|\varphi\|^2_{H^1(\mathbf{C})}.
\end{aligned}$$
On the other hand,
$$\begin{aligned}
E(\infty, 2R, \xi; \varphi) &\geq \int_{\mathbf{C}\smallsetminus\Delta(0,2R)} |\Phi_\xi \otimes \varphi|^2 \\
&\geq c_0^2 \int_{\mathbf{C}\smallsetminus\Delta(0,2R)} |\varphi|^2
\end{aligned}$$
implies
$$K_0 E(\infty, 2R, \xi; \varphi) \geq \|\varphi\|^2_{H^1(\mathbf{C}\smallsetminus\Delta(0,2R))}$$
for some $K_0 > 0$. This gives the lemma for ξ in a finite region. The case of $|\xi|$ large also follows noting that K depends at most polynomially on ξ. $\qquad\square$

Since a Δ_ξ-harmonic spinor is subharmonic in the usual sense, the above results also imply point-wise exponential decay on harmonic spinors:

LEMMA 2.32. *Suppose $R > R_0$. Then there exists $K, c > 0$ such that for any $|z| > 2R + 1$ and any Δ_ξ-harmonic spinor φ we have*
$$|\varphi(z)| \leq Ke^{-Rc}\|\varphi\|^2_{H^1(\mathbf{C})}.$$

PROOF. Because of the condition $|z| > 2R+1$, the disk $\Delta(z, 1)$ centered at z of radius 1 is contained in $\mathbf{C}\smallsetminus\Delta(0, 2R)$. On the other hand, by subharmonicity of φ with respect to the usual Laplace operator, we have
$$\begin{aligned}
|\varphi(z)| &\leq K_0 \int_{\Delta(z,1)} |\varphi(w)||\mathrm{d}w|^2 \\
&\leq K_1 \left(\int_{\Delta(z,1)} |\varphi(w)|^2 |\mathrm{d}w|^2 \right)^{1/2} \\
&\leq K_1 \left(\int_{\mathbf{C}\smallsetminus\Delta(0,2R)} |\varphi(w)|^2 |\mathrm{d}w|^2 \right)^{1/2}
\end{aligned}$$
We conclude using Lemma 2.31. $\qquad\square$

CHAPTER 3

The transform of the integrable connection

In this chapter, we define the transformed parabolic integrable connection induced by the deformation D_ξ. First, in Section 3.1, we define the underlying flat bundle; then in Section 3.2 we show that its behavior at infinity verifies appropriate asymptotic conditions. This then allows us to apply the results of [**6**] in order to define an extension into a parabolic integrable connection over the singularity at infinity; the same thing for other singularities follows from [**21**].

Before starting these points, we need however to introduce some notation. Recall first that $\hat{P}$ was defined as the set $\{\xi_1, \ldots, \xi_{n'}\}$ of eigenvalues of the second-order term of D at infinity. Let $\hat{H} \to \hat{\mathbf{C}} \smallsetminus \hat{P}$ denote the trivial Hilbert bundle with fibers $L^2(\mathbf{C}, S^- \otimes E)$. By Theorem 2.21, the transformed bundle $\hat{E}$ can be given as the vector bundle whose fiber over $\xi \in \mathbf{C} \smallsetminus \hat{P}$ is the kernel of the adjoint Dirac operator $(\dirac_\xi)^*$. By the same theorem, such an element is also Δ_ξ-harmonic. Now remark that on the bundle $\hat{H}$ there exists a hermitian metric $\langle ., . \rangle$ which is canonical once a hermitian metric $h(., .)$ is fixed on E: for any two elements $\hat{f}_1, \hat{f}_2 \in \hat{H}_\xi = L^2(\mathbf{C}, S^- \otimes E)$, it is defined by the L^2 inner product

$$\langle \hat{f}_1, \hat{f}_2 \rangle = \int_{\mathbf{C}} h(\hat{f}_1, \hat{f}_2) |\mathrm{d}z|^2.$$

Moreover, the trivial connection $\hat{\mathrm{d}}$ on the bundle $\hat{H}$ is unitary with respect to this metric. Let $\hat{\pi}_\xi$ denote orthogonal projection of $\hat{H}_\xi$ onto the subspace $\hat{E}_\xi$, and i the inclusion $\hat{E} \hookrightarrow \hat{H}$.

DEFINITION 3.1. *We call* transformed hermitian metric *the fiber metric $\hat{h}$ on $\hat{E}$ which is equal on the fiber $\hat{E}_\xi$ to the restriction of the above defined L^2 scalar product $\langle ., . \rangle$ to the subspace $\hat{E}_\xi \subset L^2(\mathbf{C}, S^- \otimes E)$.*

3.1. Construction of the transformed flat connection

In this section we show that the transformed bundle admits an integrable connection, which is determined only by the deformation D_ξ and the . First, we describe its intrinsic construction, then we give it in terms of an explicit formula.

3.1.1. Intrinsic definition. Defining a flat connection is equivalent to giving a basis of parallel sections on a disk B_0 around each point $\xi_0 \in \hat{\mathbf{C}} \smallsetminus \hat{P}$. Given this, in order to see that it defines indeed a flat connection, one only needs to prove that the transition matrices on $B_0 \cap B_1$ between two such bases (corresponding to points ξ_0 and ξ_1) are constant.

So suppose $\xi_0 \in \hat{\mathbf{C}} \smallsetminus \hat{P}$, and let $\hat{f}_1(z), \dots, \hat{f}_{\hat{r}}(z)$ be a basis of the vector space $\hat{E}_{\xi_0}$. On the basis of Lemma 2.32, for $\varepsilon_0 = \varepsilon_0(\xi_0) > 0$ sufficiently small, the expressions

$$(3.1) \qquad \hat{f}_j(\xi; z) = \hat{\pi}_\xi(e^{(\xi-\xi_0)z}\hat{f}_j(z)) \in \hat{E}_\xi$$

make sense for ξ on the ball $B_0 = B(\xi_0, \varepsilon_0)$ of radius ε_0 centered at ξ_0. Therefore, (restricting ε_0 if necessary), they define an extension of the basis $\hat{f}_1, \dots, \hat{f}_{\hat{r}}$ of the vector space $\hat{E}_{\xi_0}$ to a trivialisation of the bundle $\hat{E}$ over B_0.

PROPOSITION 3.2. *The family of sections (3.1) for all $\xi_0 \in \hat{\mathbf{C}} \smallsetminus \hat{P}$, for $j \in \{1, \dots, \hat{r}\}$, and for all $\xi \in B_0$ define a local system for a flat connection $\hat{D}$ on $\hat{E} \to \hat{\mathbf{C}} \smallsetminus \hat{P}$.*

DEFINITION 3.3. *We will call $\hat{D}$ the* transformed flat connection *on $\hat{\mathbf{C}} \smallsetminus \hat{P}$.*

PROOF. (Proposition) Let $\tilde{\xi}_0 \neq \xi_0$ be another point of $\hat{\mathbf{C}} \smallsetminus \hat{P}$, and $\hat{g}_1(z), \dots, \hat{g}_{\hat{r}}(z)$ be a basis for the vector space $\hat{E}_{\tilde{\xi}_0}$. According to (3.1), the local trivialisation of $\hat{E}$ near $\tilde{\xi}_0$ we need to consider is then $\hat{g}_1(\xi), \dots, \hat{g}_{\hat{r}}(\xi)$, with

$$(3.2) \qquad \hat{g}_l(\xi; z) = \hat{\pi}_\xi(e^{(\xi-\tilde{\xi}_0)z}\hat{g}_l(z))$$

for ξ in a small disk $\tilde{B}_0$ around $\tilde{\xi}_0$. In order to show that the local bases (3.1) and (3.2) define indeed a local system, we need to show that the transition matrices $m(\xi)$ between them are independent of the point $\xi \in B_0 \cap \tilde{B}_0$. We will make use of the following:

LEMMA 3.4. *For any $\xi, \xi' \in B_0$, and any $k_0 \in ker(D_{\xi_0}|S^- \otimes E)$ we have*

$$\hat{\pi}_{\xi'}\left(e^{(\xi'-\xi)z}\hat{\pi}_\xi(e^{(\xi-\xi_0)z}k_0(z))\right) = \hat{\pi}_{\xi'}(e^{(\xi'-\xi_0)z}k_0(z)).$$

PROOF. (Lemma) Set $k_\xi(z) = e^{(\xi-\xi_0)z}k_0(z)$; we need to prove that

$$\hat{\pi}_{\xi'}[e^{(\xi'-\xi)z}\hat{\pi}_\xi(k_\xi(z))] = \hat{\pi}_{\xi'}(e^{(\xi'-\xi)z}k_\xi(z)),$$

or equivalently that

$$\hat{\pi}_{\xi'}[e^{(\xi'-\xi)z}(\mathrm{Id} - \hat{\pi}_\xi)(k_\xi)] = 0,$$

which is still equivalent to

(3.3) $$e^{(\xi'-\xi)z}(\mathrm{Id} - \hat{\pi}_\xi)(k_\xi) \perp \hat{E}_{\xi'}.$$

Since $\hat{\pi}_\xi$ is orthogonal projection to $\hat{E}_\xi$, we have

(3.4) $$(\mathrm{Id} - \hat{\pi}_\xi)(k_\xi) \in \hat{E}_\xi^\perp.$$

Moreover, observe that for ξ_0 and ξ fixed, the relation

(3.5) $$e^{(\xi-\xi_0)z}.D_{\xi_0} = D_{\xi_0} - (\xi - \xi_0)dz\wedge = D_\xi,$$

holds, and so
(3.6)
$$k_\xi = e^{(\xi-\xi_0)z}k_0 \in e^{(\xi-\xi_0)z}ker(D_{\xi_0}) \subset ker(D_\xi) = Im(D_\xi^*)^\perp = Im(D_\xi)\oplus\hat{E}_\xi.$$

From (3.4) and (3.6) it follows that $(\mathrm{Id} - \hat{\pi}_\xi)k_\xi \in Im(D_\xi)$. Now using (3.5) for $(\xi'-\xi)$ instead of $(\xi-\xi_0)$, we deduce that $e^{(\xi'-\xi)z}(\mathrm{Id} - \hat{\pi}_\xi)k_\xi \in Im(D_{\xi'})$, whence (3.3). This finishes the proof of the lemma. □

Let us now come back to the study of the transition matrix: let $\xi, \xi' \in B_0 \cap \tilde{B}_0$, and suppose we have

(3.7) $$\hat{f}_j(\xi) = \sum_{l=1}^{\hat{r}} m_{jl}\hat{g}_l(\xi),$$

where (m_{jl}) is the transition matrix between the two bases at the point ξ. Lemma 3.4 means that for $|\xi - \xi'|$ sufficiently small, we have

(3.8) $$\hat{f}_j(\xi') = \hat{\pi}_{\xi'}(e^{(\xi'-\xi)z}\hat{f}_j(\xi))$$

(3.9) $$\hat{g}_l(\xi') = \hat{\pi}_{\xi'}(e^{(\xi'-\xi)z}\hat{g}_l(\xi)).$$

Now plugging (3.7) into (3.8), then using (3.9) we obtain

$$\hat{f}_j(\xi') = \hat{\pi}_{\xi'}\left(e^{(\xi'-\xi)z}\sum_{l=1}^{\hat{r}}m_{jl}\hat{g}_l(\xi)\right)$$
$$= \sum_{l=1}^{\hat{r}}m_{jl}\hat{\pi}_{\xi'}(e^{(\xi'-\xi)z}\hat{g}_l(\xi))$$
$$= \sum_{l=1}^{\hat{r}}m_{jl}\hat{g}_l(\xi'),$$

so the transition matrix at the point ξ' is the same as the one at ξ, whence we obtain the Proposition. $\qquad\square$

3.1.2. Explicit description. We now give an explicit formula for the flat connection constructed above. In the sequel we follow [**15**]. First define a unitary connection on $\hat{E}$ with respect to the transformed hermitian metric by

(3.10) $$\hat{\nabla} = \hat{\pi}_\xi \circ \hat{\mathrm{d}} \circ i.$$

The fact that this connection is indeed $\hat{h}$-unitary can be seen as follows: let $f, g \in \Gamma(\hat{E})$ be local sections around ξ_0, then from orthogonality of $\hat{\pi}_\xi$ to $\hat{E}$ with respect to the norm $\langle.,.\rangle$ we have in ξ_0

$$\hat{\mathrm{d}}(\hat{h}(\hat{f},\hat{g})) = \hat{\mathrm{d}}\langle\hat{f},\hat{g}\rangle = \langle\hat{\mathrm{d}}\hat{f},\hat{g}\rangle + \langle\hat{f},\hat{\mathrm{d}}\hat{g}\rangle$$
$$= \langle\hat{\nabla}\hat{f},\hat{g}\rangle + \langle\hat{f},\hat{\nabla}\hat{g}\rangle = \hat{h}(\hat{\nabla}\hat{f},\hat{g}) + \hat{h}(\hat{f},\hat{\nabla}\hat{g}),$$

where $\hat{\mathrm{d}}$ stands for exterior differentiation of functions along the coordinate ξ as well as for the trivial connection with respect to ξ on the trivial Hilbert bundle $\hat{H}$. Finally, we define an endomorphism-valued $(1,0)$-form (a candidate to be a transformed Higgs field) by mapping a Δ_ξ-harmonic section $\hat{f}(\xi; z)$ to

(3.11) $$\hat{\theta}_\xi(\hat{f}(\xi; z)) = -\frac{1}{2}\hat{\pi}_\xi(z\hat{f}(\xi; z))\mathrm{d}\xi,$$

where $\mathrm{d}\xi$ stands for the standard generator of the holomorphic $(1,0)$-forms on $\hat{\mathbf{C}}$. This field will indeed be holomorphic provided that the original metric h is harmonic (see Section 4.2).

PROPOSITION 3.5. *The connection $\hat{\nabla} + 2\hat{\theta}$ is equal to the transformed flat connection $\hat{D}$ defined above.*

PROOF. We need to show that for all ξ_0 and all $f(z) \in \hat{E}_{\xi_0}$, the local $\hat{D}$-parallel section in $\xi \in B_0$ given by

$$(3.12) \qquad \hat{f}(\xi; z) = \hat{\pi}_\xi(e^{(\xi - \xi_0)z} \hat{f}(z))$$

is parallel in B_0 with respect to $\hat{\nabla} + 2\hat{\theta}$. First, let us check it in ξ_0:

$$((\hat{\nabla} + 2\hat{\theta})\hat{f})(\xi_0) = \hat{\pi}_{\xi_0}[(\hat{d}\hat{f})(\xi_0) - z\hat{f}(\xi_0)d\xi].$$

We observe that by (3.12) we have

$$(\hat{d}\hat{f})(\xi_0) = (\hat{d}\hat{\pi}_\xi)_{\xi_0}\hat{f}(\xi_0) + \hat{\pi}_{\xi_0}(z\hat{f}(\xi_0)d\xi),$$

hence

$$((\hat{\nabla} + 2\hat{\theta})\hat{f})(\xi_0) = \hat{\pi}_{\xi_0}[(\hat{d}\hat{\pi}_\xi)_{\xi_0}\hat{f}(\xi_0)].$$

Now $\hat{\pi}_\xi \circ \hat{\pi}_\xi = \hat{\pi}_\xi$ implies

$$\hat{d}\hat{\pi}_\xi \circ \hat{\pi}_\xi + \hat{\pi}_\xi \circ \hat{d}\hat{\pi}_\xi = \hat{d}\hat{\pi}_\xi,$$

therefore

$$\hat{\pi}_{\xi_0}[(\hat{d}\hat{\pi}_\xi)_{\xi_0}\hat{f}(\xi_0)] = (\hat{d}\hat{\pi}_\xi)_{\xi_0} \circ (\mathrm{Id} - \hat{\pi}_{\xi_0})\hat{f}(\xi_0) = 0,$$

since $\hat{\pi}_{\xi_0}$ is the projection to $\hat{E}_{\xi_0}$ and $\hat{f}(\xi_0) \in \hat{E}_{\xi_0}$.

Next, fix an arbitrary $\xi \in B_0$. Then, as we have just shown, the local section defined for $|\xi' - \xi|$ sufficiently small by

$$\hat{f}'(\xi') = \hat{\pi}_{\xi'}(e^{(\xi' - \xi)z}\hat{f}(\xi; z))$$

is parallel in ξ (compare with (3.12), setting $\xi_0 = \xi, \xi = \xi'$). But Lemma 3.4 tells us that the local sections $\hat{f}'$ and $\hat{f}$ coincide in a neighborhood of ξ; in particular $\hat{f}$ is parallel in ξ. $\square$

The following is now immediate:

PROPOSITION 3.6. *The unitary part of the transformed flat connection $\hat{D}$ is*

$$\hat{D}^+ = \hat{\nabla} + \hat{\theta} - \hat{\theta}^* = \hat{\pi}_\xi \circ (\hat{d} - \frac{1}{2}z d\xi \wedge + \frac{1}{2}\bar{z} d\bar{\xi} \wedge).$$

DEFINITION 3.7. *We will call the above unitary connection $\hat{D}^+$ the transformed unitary connection. The covariant derivative associated to it will be denoted $\hat{\nabla}^+$.*

REMARK 3.8. *The fact that the formula for the transformed unitary connection involves extra multiplication terms by z and $\bar{z}$ compared to the usual formulae of other Nahm transforms is an artifact: as we will see in the next chapter, the transform admits an interpretation from the point of view of Higgs bundles, in which the formula for the transformed unitary connection agrees with the usual one.*

3.2. Extension over the singularities

At this point, it should be pointed out that a priori we have no guarantee that the constructed flat connection is indeed of the form given in [**6**] (and therefore extends nicely over the singularities); that is, in an orthonormal basis with respect to its harmonic metric it is not necessarily the model (1.20) up to a perturbation described in (2.11) and (2.13). However, there is a theorem of O. Biquard and M. Jardim which allows us to show that this is the case. Namely, Theorem 0.1 of [**7**] states the following:

THEOREM 3.9. *Let $\tilde{A}$ be an $SU(2)$-instanton on $\mathbf{R}^4$, invariant with respect to the additive subgroup $\mathbf{Z}\frac{\partial}{\partial x_3} \oplus \mathbf{Z}\frac{\partial}{\partial x_4}$, and suppose that its curvature $F_{\tilde{A}}$ has quadratic decay at infinity (that is, $|F_{\tilde{A}}| = O(r^{-2})$, where $r^2 = x_1^2 + x_2^2$). Then there exists a gauge near infinity in which $\tilde{A}$ is asymptotic to the following model:*

$$\tilde{A}_0 = d + i\Big(\lambda_1 \, dx_3 + \lambda_2 \, dx_4 + (\mu_1 \cos\theta - \mu_2 \sin\theta)\frac{dx_3}{r}$$
$$+ (\mu_1 \sin\theta + \mu_2 \cos\theta)\frac{dx_4}{r} + \alpha \, d\theta\Big),$$

where $z = re^{i\theta}$ are coordinates for the (x_1, x_2)-plane. Moreover, the difference a between $\tilde{A}$ and this model satisfies

$$|a| = O(r^{-1-\delta}), \quad |\nabla_{\tilde{A}_0} a| = O(r^{-2-\delta}).$$

In order to be able to apply this result to our case, consider the Euclidean space $(\mathbf{R}^4)^*$ spanned by orthonormal vectors $\frac{\partial}{\partial x_j^*}$ for $j = 1, 2, 3, 4$, and identify the subspace spanned by $\frac{\partial}{\partial x_1^*}$ and $\frac{\partial}{\partial x_2^*}$ with the line $\hat{\mathbf{C}}$ with complex coordinate ξ underlying $\hat{\mathrm{D}}$. By the results of [**12**], $\hat{D}$ then induces an instanton $\tilde{A}$ on $(\mathbf{R}^4)^*$ with singularities, invariant with respect to the subspace $\mathbf{R}\frac{\partial}{\partial x_3^*} \oplus \mathbf{R}\frac{\partial}{\partial x_4^*}$. In particular, $\tilde{A}$ is invariant with respect to $\mathbf{Z}\frac{\partial}{\partial x_3^*} \oplus \mathbf{Z}\frac{\partial}{\partial x_4^*}$, so Theorem 3.9 can be applied to it, provided

that its curvature has quadratic decay. In order to have an explicit description of $\tilde{A}$ and its curvature, remember that $\hat{D}$ decomposes as

$$\hat{D} = \hat{\nabla}^+ + \hat{\theta} + \hat{\theta}^*,$$

where $\hat{\nabla}^+$ is the transformed unitary connection, $\hat{\theta}$ the field defined in (3.11) and $\hat{\theta}^*$ its adjoint with respect to the harmonic metric of $\hat{D}$. Now as we will see in Section 4.2, this harmonic metric is in fact the transformed hermitian metric $\hat{h}$ given in Definition 3.1. The unitary part of $\hat{D}$ decomposes further into its $(1,0)$- and $(0,1)$-part:

$$\hat{\nabla}^+ = (\hat{\nabla}^+)^{1,0} + (\hat{\nabla}^+)^{0,1}.$$

Finally, we write $\hat{\vartheta}$ for the endomorphism-part of $\hat{\theta}$:

$$\hat{\theta} = \hat{\vartheta}\mathrm{d}\xi.$$

The instanton over $(\mathbf{R}^4)^*$ corresponding to $\hat{D}$ is then given by the formula

$$\tilde{A} = \hat{\nabla}^+ + \Re\hat{\vartheta}\mathrm{d}x_3^* + \Im\hat{\vartheta}\mathrm{d}x_4^*,$$

where we recall that

$$\frac{\partial}{\partial\xi} = \frac{1}{2}\left(\frac{\partial}{\partial x_1^*} - \frac{\partial}{\partial x_2^*}\right)$$

is the natural complex coordinate of $\hat{\mathbf{C}}$, and the connection $\hat{\nabla}^+$ on $(\mathbf{R}^4)^*$ acts as $\hat{\nabla}^+$ along $\hat{\mathbf{C}}$ and as the trivial connection along $\mathbf{R}\frac{\partial}{\partial x_3^*} \oplus \mathbf{R}\frac{\partial}{\partial x_4^*}$. Furthermore, as it can be seen from the results in Section 1 of [**12**], we then have the formula

$$\begin{aligned}
F_{\tilde{A}} = &-[\hat{\vartheta}, \hat{\vartheta}^*](\mathrm{d}x_1^* \wedge \mathrm{d}x_2^* + \mathrm{d}x_3^* \wedge \mathrm{d}x_4^*) \\
&+ (\hat{\nabla}^+)_{x_1^*}\Re\hat{\vartheta}(\mathrm{d}x_1^* \wedge \mathrm{d}x_3^* - \mathrm{d}x_2^* \wedge \mathrm{d}x_4^*) \\
&+ (\hat{\nabla}^+)_{x_1^*}\Im\hat{\vartheta}(\mathrm{d}x_1^* \wedge \mathrm{d}x_4^* + \mathrm{d}x_2^* \wedge \mathrm{d}x_3^*),
\end{aligned}$$

(3.13)

where we have written $(\hat{\nabla}^+)_{x^*}$ to denote the action of the unitary connection in the $\frac{\partial}{\partial x^*}$-direction. Hence, before we can apply Theorem 3.9 we need to check the following:

THEOREM 3.10. *There exists a constant $K > 0$ such that the commutator $[\hat{\vartheta}, \hat{\vartheta}^*]$ is bounded by $K|\xi|^{-2}$ as $\xi \to \infty$. The same estimation holds for $\hat{\nabla}^+\hat{\vartheta}$.*

PROOF. We start with the case of the commutator. Let $\hat{f}(\xi; z) \in \hat{E}_\xi = Ker(\hat{\phi}_\xi)^*$ be arbitrary; we wish to show the estimate

$$\left| [\hat{\vartheta}, \hat{\vartheta}^*]\hat{f}(\xi) \right|_{\hat{h}} \leq K|\xi|^{-2}|\hat{f}(\xi)|_{\hat{h}},$$

with K independent of $\hat{f}$ and of ξ. Recall the well-known formula from Hodge theory:

(3.14)
$$\hat{\pi}_\xi = \mathrm{Id} - \hat{\phi}_\xi G_\xi \hat{\phi}_\xi^*.$$

Using this, we obtain

$$[\hat{\vartheta}, \hat{\vartheta}^*]\hat{f}(\xi) = -\frac{1}{2}\hat{\pi}_\xi(z\hat{\pi}_\xi(\bar{z}\hat{f}(\xi)) - \bar{z}\hat{\pi}_\xi(z\hat{f}(\xi)))$$

(3.15)
$$= \frac{1}{2}\hat{\pi}_\xi(z\hat{\phi}_\xi G_\xi \hat{\phi}_\xi^*(\bar{z}\hat{f}(\xi)) - \bar{z}\hat{\phi}_\xi G_\xi \hat{\phi}_\xi^*(z\hat{f}(\xi))).$$

Since D_ξ is a connection, the following commutation relations hold:

$$[D_\xi, z] = \mathrm{d}z\wedge \qquad\qquad [D_\xi, \bar{z}] = \mathrm{d}\bar{z}\wedge$$

$$[D_\xi^*, z] = \frac{\partial}{\partial\bar{z}}\llcorner \qquad\qquad [D_\xi^*, \bar{z}] = \frac{\partial}{\partial z},$$

where $\llcorner$ stands for contraction of a differential form by a vector field. It follows immediately

(3.16)
$$[\hat{\phi}_\xi, z] = -[\hat{\phi}_\xi^*, z] = \mathrm{d}z \wedge -\frac{\partial}{\partial\bar{z}}\llcorner = \mathrm{d}z\cdot$$

(3.17)
$$[\hat{\phi}_\xi, \bar{z}] = -[\hat{\phi}_\xi^*, \bar{z}] = \mathrm{d}\bar{z} \wedge -\frac{\partial}{\partial z}\llcorner = \mathrm{d}\bar{z}\cdot$$

where the Clifford multiplication $\cdot$ is defined by these formulae. Plugging these in the expression (3.15), using $\hat{\phi}_\xi^*\hat{f}(\xi; z) = 0$ and $\hat{\pi}_\xi|_{Im\hat{\phi}_\xi^*} = 0$ together with the definition of $\hat{h}$, we get

$$\left| [\hat{\vartheta}, \hat{\vartheta}^*]\hat{f}(\xi) \right|_{\hat{h}} = \frac{1}{2}\left\| \hat{\pi}_\xi\left(\mathrm{d}z \cdot G_\xi \mathrm{d}\bar{z} \cdot \hat{f}(\xi) - \mathrm{d}\bar{z} \cdot G_\xi \mathrm{d}z \cdot \hat{f}(\xi)\right) \right\|_{L^2(\mathbf{C})}$$

(3.18)
$$\leq \frac{1}{2}\left\| G_\xi \mathrm{d}\bar{z} \cdot \hat{f}(\xi) \right\|_{L^2(\mathbf{C})} + \frac{1}{2}\left\| G_\xi \mathrm{d}z \cdot \hat{f}(\xi) \right\|_{L^2(\mathbf{C})},$$

since the norm of the orthogonal projection of a vector to a subspace is at most the norm of the vector and the action of Clifford multiplication by $\mathrm{d}z$ and $\mathrm{d}\bar{z}$ is point-wise bounded. We conclude by the first statement of Lemma 2.25.

Next, let us come to $\hat{\nabla}^+\hat{\vartheta}$. Similarly to the above, using (3.14) and the commutation formulae (3.16)-(3.17) we obtain

$$\left(\hat{\nabla}^+\hat{\vartheta}\right)\hat{f}(\xi) = \left(\hat{D}^+ \circ \hat{\vartheta} - \hat{\vartheta} \circ \hat{D}^+\right)\hat{f}(\xi)$$

$$=\hat{\pi}_\xi\left(\hat{d} - \frac{z}{2}d\xi + \frac{\bar{z}}{2}d\bar{\xi}\right)\hat{\pi}_\xi\left(-\frac{z}{2}\right)\hat{f}(\xi)$$

$$- \hat{\pi}_\xi\left(-\frac{z}{2}\right)\hat{\pi}_\xi\left(\hat{d} - \frac{z}{2}d\xi + \frac{\bar{z}}{2}d\bar{\xi}\right)\hat{f}(\xi)$$

$$=\hat{\pi}_\xi\left[\left(\hat{d} - \frac{z}{2}d\xi + \frac{\bar{z}}{2}d\bar{\xi}\right)\phi_\xi G_\xi \phi_\xi^*\left(\frac{z}{2}\hat{f}(\xi)\right)\right.$$

$$\left.- \frac{z}{2}\phi_\xi G_\xi \phi_\xi^*\left(\hat{d} - \frac{z}{2}d\xi + \frac{\bar{z}}{2}d\bar{\xi}\right)\hat{f}(\xi)\right]$$

$$=\hat{\pi}_\xi\left[\left(\frac{1}{2}d\xi \wedge dz - \frac{1}{2}d\bar{\xi} \wedge d\bar{z}\right) \cdot G_\xi \frac{dz}{2} \cdot \hat{f}(\xi)\right.$$

$$\left.- \frac{dz}{2} \cdot G_\xi \left(\frac{1}{2}d\xi \wedge dz - \frac{1}{2}d\bar{\xi} \wedge d\bar{z}\right) \cdot \hat{f}(\xi)\right]$$

$$+ \hat{\pi}_\xi\left[\hat{d}\phi_\xi G_\xi \frac{dz}{2} \cdot \hat{f}(\xi) - \frac{dz}{2} \cdot G_\xi \phi_\xi^*\hat{d}\hat{f}(\xi)\right]$$

(here dz and $d\bar{z}$ act on the spinors by Clifford multiplication, whereas $d\xi$ and $d\bar{\xi}$ by wedge product). Noticing that $|d\xi| = |d\bar{\xi}| = 2$, the first term in the last expression can be treated exactly as in (3.18). For the second term, one only needs to remark that the commutation relations

$$\left[\hat{d}, D_\xi\right] = \left[\hat{d}, D - \frac{\xi}{2}dz + \frac{\bar{\xi}}{2}d\bar{z}\right]$$

$$= -\frac{d\xi \wedge dz\wedge}{2} + \frac{d\bar{\xi} \wedge d\bar{z}\wedge}{2}$$

and

$$\left[\hat{d}, D_\xi^*\right] = -\frac{d\xi\wedge}{2}\frac{\partial}{\partial\bar{z}}\llcorner + \frac{d\bar{\xi}\wedge}{2}\frac{\partial}{\partial z}\llcorner$$

show that

$$\left[\hat{d}, \phi_\xi\right] = -\left[\hat{d}, \phi_\xi^*\right] = -\frac{1}{2}d\xi \wedge dz \cdot + \frac{1}{2}d\bar{\xi} \wedge d\bar{z}\cdot$$

holds. Therefore we can proceed again as in (3.18).   □

On the basis of Theorem 3.9, the behavior of the transformed flat connection at infinity satisfies the hypothesis considered in [**6**]. Namely, in a suitable gauge its difference from a model with second-order pole is in the weighted Sobolev-space $L^{1,2}_{-2+\delta}(\Omega^1 \otimes E)$ considered in Section 2 of that article. Indeed, passing to a coordinate $w = z^{-1}$, $|w| = \rho$ in which the double pole is in 0, the norm of the perturbation is $O(\rho^{1+\delta})$,

whereas that of its derivative is also $O(\rho^{1+\delta})$ (because the norm of 1-forms near infinity is $|\mathrm{d}z| = |\mathrm{d}w|/|w| = 1$), and we conclude since $\rho^{1+\delta}/\rho^2 \in L^2_{\delta-2}$. It follows from the results of its Sections 7 and 8 that the analytic flat connection $\hat{D}$ defined outside infinity extends to an algebraic integrable connection with a parabolic structure on the singular fiber at infinity. On the other hand, such an extension over logarithmic singularities (that is, singularities in which the eigenvalues of $\hat{D}$ or equivalently those of $\hat{\vartheta}$ have at most first-order poles) is ensured by Theorem 2 of [21]. Therefore, by Theorem 4.30 the flat connection $\hat{D}$ on $\hat{\mathbf{C}} \smallsetminus \hat{P}$ can be extended into a meromorphic integrable connection on $\widehat{\mathbf{CP}}^1$ with parabolic structures at the singularities.

DEFINITION 3.11. *The* transformed meromorphic integrable connection *is the meromorphic integrable connection with parabolic structure in the singularities induced by the above extension procedures, subject to local changes of holomorphic trivialisations near the singularities to take all weights between 0 and 1. We will continue to denote it by* $(\hat{E}, \hat{D})$. *The underlying extension will be called* transformed extension *of the transformed bundle.*

REMARK 3.12. *We will see in Section 4.6 that the parabolic structures are adapted to the harmonic metric; namely, the weight $0 \leq \hat{\alpha}_k < 1$ of a subspace $F_k \hat{E}|_p$ of a singular fiber corresponds in local coordinate z vanishing at the puncture to a decay bounded above by $|z|^{2\hat{\alpha}_k}$ of the norm of a parallel section extending an element of $F_k \hat{E}|_p$, as measured by the harmonic metric. However, in Sections 4.4.1 and 4.4.2 we will construct a different extension over the punctures – more suited to analytical study –, where the behavior of the norm of parallel sections near the singular points will no longer be bounded. We then pass back to the transformed extension in Corollary 4.39, where we remark that it is the one that establishes a "good" correspondence.*

CHAPTER 4

Interpretation from the point of view of Higgs bundles

Let (E, D, h) be a Hermitian bundle with integrable connection. Throughout this chapter, we suppose that the original metric h is harmonic. This metric then defines a Higgs bundle $(\mathcal{E}, \theta)$ starting from the integrable connection, via the procedure described in Section 1.6. We first prove that the transformed metric $\hat{h}$ is then harmonic for $\hat{D}$. Next, we give an interpretation of the transformed Higgs bundle of $(\mathcal{E}, \theta)$ in terms of the hypercohomology of a sheaf map over $\mathbf{CP}^1$. These results will then be used to define the *induced extension* ${}^i\hat{\mathcal{E}}$ of the transformed bundle over the punctures $\hat{P} \cup \{\infty\}$, and to compute the topology and the singularity parameters of this extension of the transformed Higgs bundle. This will enable us to eventually compute the topology and the singularity parameters of the transformed Higgs bundle with respect to its transformed extension given in Definition 3.11.

4.1. The link with the transformed integrable connection

Recall that we have defined the deformation of the Higgs bundle by the formula (1.35), and we write D''_ξ for the D''-operator of this deformation. Explicitly, we have

$$D''_\xi = D'' + \theta_\xi,$$

where $\theta_\xi = \theta - \xi/2 \mathrm{d}z$. Moreover, as we have noticed in Section 1.8, nonabelian Hodge theory identifies the deformation of the Higgs bundle structure (1.35) and that of the integrable connection via the unitary gauge transformation

$$g(z, \xi) = e^{[\bar{\xi}\bar{z} - \xi z]/2}.$$

In other words, writing $g_\xi = g(., \xi)$ for the gauge transformation restricted to the fiber $\hat{H}_\xi$, we have

$$(4.1) \qquad g_\xi . D_\xi = D^H_\xi = D - \frac{\xi}{2}\mathrm{d}z \wedge -\frac{\bar{\xi}}{2}\mathrm{d}\bar{z} \wedge .$$

Since the gauge transformation g_ξ is unitary, in addition to (4.1) we have as well

$$(4.2) \qquad\qquad g_\xi.D_\xi^* = (D_\xi^H)^*.$$

DEFINITION 4.1. *The operator $\slashed\partial_\xi^H = D_\xi^H - (D_\xi^H)^*$ will be referred to as the* Higgs Dirac operator. *In the same way, we let $\slashed\partial_\xi''$ stand for the Dirac operator $D_\xi'' - (D_\xi'')^*$. The transformed smooth bundle underlying the Higgs bundle is the bundle $\hat V$ over $\hat{\mathbf C} \setminus \hat P$ whose fiber over ξ is the first L^2-cohomology space $L^2 H^1(\mathcal C_\xi^H)$ of the operator D_ξ^H.*

PROPOSITION 4.2. *This way we define a smooth vector bundle $\hat V$. Furthermore, there exists a canonical bundle isomorphism between the smooth bundle $\hat E$ underlying the transformed integrable connection and the smooth bundle $\hat V$ underlying the transformed Higgs bundle.*

PROOF. Theorem 2.16 tells us that the transformed bundle underlying the integrable connection is the bundle of first L^2-cohomologies of D_ξ^{int}. For any ξ, the gauge transformation g_ξ of E induces a natural isomorphism between the L^2-cohomology spaces of the complexes (1.7) and

$$(4.3) \qquad \Omega^0 \otimes E \xrightarrow{g_\xi.D_\xi^{int}} \Omega^1 \otimes E \xrightarrow{g_\xi.D_\xi^{int}} \Omega^2 \otimes E.$$

which is just $\mathcal C_\xi^H$. In Theorem 2.6 we have shown that the 0-th and 2-nd cohomology of $\mathcal C_\xi$ vanishes for all $\xi \in \hat{\mathbf C} \setminus \hat P$, whereas Corollary 2.7 implies that the cohomology spaces $L^2 H^1(\mathcal C_\xi)$ define a smooth vector bundle over $\hat{\mathbf C} \setminus \hat P$. This then implies the same thing for $\mathcal C_\xi^H$, whence the bundle isomorphism between the bundles over $\hat{\mathbf C} \setminus \hat P$ in question. $\square$

Theorem 2.21 has the following interpretation:

THEOREM 4.3. *The first L^2-cohomology $\hat V_\xi = L^2 H^1(\mathcal C_\xi^H)$ of the operator D_ξ^H is canonically isomorphic to the kernel of the adjoint Dirac operator*

$$(4.4) \qquad (\slashed\partial_\xi^H)^* : L^2(S^- \otimes E) \longrightarrow L^2(S^+ \otimes E)$$

on its domain, or alternatively to the kernel of the Laplace operator

$$(4.5) \qquad \Delta_\xi^H = \slashed\partial_\xi^H (\slashed\partial_\xi^H)^* : L^2(S^- \otimes E) \longrightarrow L^2(S^- \otimes E)$$

on its domain.

PROOF. Apply the gauge transformation g to Theorem 2.21 and notice that (4.1) and (4.2) imply

(4.6) $$g_\xi . \hat{\phi}_\xi^* = (\hat{\phi}_\xi^H)^*$$

and

(4.7) $$g_\xi . \Delta_\xi = \Delta_\xi^H;$$

and in particular that

(4.8) $$g_\xi(Ker(\hat{\phi}_\xi^*)) = Ker((\hat{\phi}_\xi^H)^*)$$

and

(4.9) $$g_\xi(Ker(\Delta_\xi)) = Ker(\Delta_\xi^H).$$

$\square$

This result enables us to put similar definitions as in the integrable deformation case.

DEFINITION 4.4. *The hermitian bundle metric on $\hat{V}$ given by L^2 scalar product of the $(\hat{\phi}_\xi^H)^*$-harmonic representative will be called the transformed hermitian metric, and will be denoted by $\hat{h}$. Also, $\hat{\pi}_\xi^H$ will stand for $\hat{h}$-orthogonal projection of $L^2(S^- \otimes E)$ onto $\hat{V}$.*

REMARK 4.5. *Starting from a Higgs bundle with any Hermitian metric (not necessary harmonic), we can define in the same way its transform on the transformed bundle $\hat{V}$.*

Next, we recollect the above considerations in terms of the transformed bundles.

PROPOSITION 4.6. *The family of gauge transformations g induce a Hermitian bundle isomorphism between $\hat{E}$ and $\hat{V}$. Furthermore, the fiber $\hat{V}_\xi$ can be identified with the first L^2-cohomology of the single complex associated to the following double complex, denoted by $\mathcal{D}_\xi$:*

$$
\begin{array}{ccc}
\Omega^{0,1} \otimes E & \xrightarrow{\ \theta_\xi \wedge\ } & \Omega^2 \otimes E \\
{\scriptstyle \bar{\partial}^\varepsilon} \uparrow & & \uparrow {\scriptstyle \bar{\partial}^\varepsilon} \\
\Omega^0 \otimes E & \xrightarrow{\ \theta_\xi \wedge\ } & \Omega^{1,0} \otimes E.
\end{array}
$$

REMARK 4.7. *Notice that commutativity of this diagram follows from the hypothesis $\bar{\partial}^\varepsilon \theta = 0$, which is just the definition of the harmonicity of h.*

PROOF. By (4.9),the D_ξ-harmonic representative of a class is mapped by g into a D_ξ^H-harmonic class. Since the transformed metric from both points of view is induced by L^2-norm of the harmonic representatives, and g is unitary, this gives the first statement. For the second, remark that by Theorem 1.25, the Laplace operator Δ_ξ^H is equal (up to a factor of 2) to the Laplace operator $\Delta_\xi'' = \cancel{\partial}_\xi''(\cancel{\partial}_\xi'')^*$, therefore their kernels coincide. This then identifies $\hat{V}$ with the first L^2-cohomology of the complex

$$(4.10) \qquad \Omega^0 \otimes E \xrightarrow{D_\xi''} \Omega^1 \otimes E \xrightarrow{D_\xi''} \Omega^2 \otimes E.$$

Finally, recall that the formula

$$D_\xi'' = \bar{\partial}^{\mathcal{E}} + \theta_\xi$$

gives the decomposition of D_ξ'' into its $(0,1)$- and $(1,0)$-part respectively. This means that the complex (4.10) is the single complex associated to the double complex $\mathcal{D}_\xi$. However, it is not necessarily true that the domain of D_ξ'' is the sum of the domain of $\bar{\partial}^{\mathcal{E}}$ and that of θ_ξ, it could in principle be larger. Still, the two L^2-cohomologies are the same. Indeed, suppose $f = f^{1,0}dz + f^{0,1}d\bar{z} \in L^2(\Omega^1 \otimes E)$ is in the kernel of D_ξ'', that is

$$(4.11) \qquad \bar{\partial}^{\mathcal{E}} f^{1,0}dz + \theta_\xi \wedge f^{0,1}d\bar{z} = 0.$$

We wish to represent the D_ξ''-cohomology class of f by a class $\tilde{f}^{1,0}dz + \tilde{f}^{0,1}d\bar{z}$ such that $\bar{\partial}\tilde{f}^{1,0} \in L^2$ and $\theta_\xi\tilde{f}^{0,1} \in L^2$. Away from logarithmic singularities, one can simply choose f itself, for there locally $f^{0,1} \in L^2$ implies $\theta_\xi f^{0,1} \in L^2$ and by (4.11) then $\bar{\partial}^{\mathcal{E}} f^{1,0} \in L^2$ as well. Thus we only need to modify f in a neighborhood of the logarithmic punctures. By Claim 4.11 near any such puncture we can find $g \in L^2(E)$ such that $\theta_\xi g \in L^2(\Omega^{1,0} \otimes E)$ and

$$f^{0,1}d\bar{z} + \bar{\partial}^{\mathcal{E}} g = 0.$$

Using $\bar{\partial}^{\mathcal{E}} \theta_\xi = 0$, the last two identities then also imply

$$\bar{\partial}^{\mathcal{E}}(f^{1,0}dz + \theta_\xi g) = 0.$$

Put $\tilde{f}^{1,0}dz = f^{1,0}dz + \theta_\xi g$; as both $f^{1,0}$ and $\theta_\xi g$ are supposed to be in L^2, so is $\tilde{f}^{1,0}dz$. This then shows that f is cohomologuos in the L^2 complex of (4.10) to a class locally represented by a section $\tilde{f}^{1,0}dz$, where $\tilde{f}^{1,0} \in L^2$ and $\bar{\partial}^{\mathcal{E}} \tilde{f}^{1,0} \in L^2$. In different terms $\tilde{f}^{1,0}dz \in \text{Dom}_{\max}(\bar{\partial}^{\mathcal{E}})$, and this

shows that the first L^2-cohomology of (4.10) is indeed equal to that of $\mathcal{D}_\xi$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Next, let us investigate what the transformed integrable connection $\hat{D}$ and its unitary part $\hat{D}^+$ become under this gauge transformation. Notice that since the gauge transformation g is unitary, the orthogonal projection $\hat{\pi}$ onto $\hat{E}$ is transformed into the orthogonal projection $\hat{\pi}^H$ onto $\hat{V}$, with respect to the same L^2-metric on the fibers; in different terms $g_\xi.\hat{\pi}_\xi = \hat{\pi}_\xi^H$. The image of the transformed integrable connection $\hat{D}$ under the gauge transformation g in the point ξ is given by

$$
\begin{aligned}
\hat{D}^H &= g.\hat{D} \\
&= g.(\hat{\pi}_\xi \circ (\hat{\mathrm{d}} - z\mathrm{d}\xi\wedge)) \\
&= \hat{\pi}_\xi^H \left(\hat{\mathrm{d}} - \frac{1}{2}(z\mathrm{d}\xi \wedge +\bar{z}\mathrm{d}\bar{\xi}\wedge) \right),
\end{aligned}
$$
(4.12)

(see (3.10), (3.11) and Proposition 3.5), and that of the candidate Higgs field is the endomorphism

$$
\begin{aligned}
\hat{\theta}^H &= g.\hat{\theta} \\
&= g.(\hat{\pi}_\xi \circ (-z/2\mathrm{d}\xi\wedge)) \\
&= -\frac{1}{2}\hat{\pi}_\xi^H(z\mathrm{d}\xi\wedge).
\end{aligned}
$$
(4.13)

Therefore, if we decompose the transformed flat connection in the point of view of Higgs bundles into its unitary and self-adjoint part, we obtain

$$
(4.14) \qquad (\hat{D}^H)^+ = \hat{\pi}_\xi^H(\hat{\mathrm{d}}) \qquad\qquad (\hat{D}^H)^{sa} = \hat{\theta}^H + (\hat{\theta}^H)^*
$$

(these formulae can also be deduced directly from Proposition 3.6). This then gives the desired interpretation of the transformed unitary connection $\hat{D}^+$ in this point of view.

DEFINITION 4.8. *We let $\bar{\partial}^{\hat{\mathcal{E}}}$ stand for the $(0,1)$-part of $(\hat{D}^H)^+$. Moreover, we call the holomorphic bundle $\hat{V}$ with partial connection $\bar{\partial}^{\hat{\mathcal{E}}}$ the transformed holomorphic bundle and we denote it by $\hat{\mathcal{E}}$.*

4.2. Harmonicity of the transformed metric

In this section we prove the following result:

THEOREM 4.9. *If the original metric h is harmonic, then the same thing is true for $\hat{h}$.*

PROOF. First remark that by (4.14), the formula for $\bar{\partial}^{\hat{\mathcal{E}}}$ is $\hat{\pi}_\xi^H(\hat{\mathrm{d}}^{0,1})$. Also, the $(1,0)$-part of $(\hat{D}^H)^{sa}$ is just $\hat{\theta}^H$. By definition, harmonicity of the transformed metric $\hat{h}$ resumes then in the equation

(4.15) $$\bar{\partial}^{\hat{\mathcal{E}}}\hat{\theta}^H = 0.$$

By Proposition 4.6 we have $\hat{V}_\xi = L^2 H^1(D''_\xi)$, with $D''_\xi = D'' - \xi/2\mathrm{d}z$. From this formula it is clear that D''_ξ depends holomorphically on ξ, so we are in the situation described in part 3.1.3 of [**10**] of chain complexes

$$\Omega^0 \otimes E \xrightarrow{D''_\xi} \Omega^1 \otimes E \xrightarrow{D''_\xi} \Omega^2 \otimes E$$

varying holomorphically with ξ. There it is shown that if the first cohomology spaces $\hat{V}_\xi$ of these complexes are all finite dimensional, of the same dimension, then the bundle $\hat{V}$ constructed out of them over the parameter space of ξ carries a natural holomorphic structure. Explicitly, this is given by by saying that a section $f \in \Gamma(\hat{V})$ in a neighborhood of ξ_0 is holomorphic if and only if it admits a lift $\tilde{f} \in \Gamma(Ker(D''_\xi|_{\Omega^1}))$ which is itself holomorphic with respect to the holomorphic structure induced by the $(0,1)$-part $\hat{\mathrm{d}}^{0,1}$ of the trivial connection $\hat{\mathrm{d}}$ on the Hilbert bundle $\hat{H}$. This holomorphic structure is the same as the one defined by the operator $\bar{\partial}^{\hat{\mathcal{E}}}$, since both are induced by $\hat{\mathrm{d}}^{0,1}$ and $\hat{\pi}^H$. The section $\hat{\theta}^H \in End(\hat{V}) \otimes \Omega_{\hat{\mathbf{C}}}^{1,0}$ is then holomorphic for this holomorphic structure if and only if it maps each holomorphic section f into a holomorphic section. In particular, this is the case if it admits a lift

$$
\begin{array}{ccc}
Ker(D''_\xi|_{\Omega^1}) & \xrightarrow{\Theta} & Ker(D''_\xi|_{\Omega^1}) \otimes \Omega_{\hat{\mathbf{C}}}^{1,0} \\
\uparrow & & \uparrow \\
\hat{V}_\xi & \xrightarrow{\hat{\theta}^H} & \hat{V}_\xi \otimes \Omega_{\hat{\mathbf{C}}}^{1,0},
\end{array}
$$

such that

(1) Θ passes to the quotient $Ker(D''_\xi|_{\Omega^1}) \to Ker(D''_\xi|_{\Omega^1})/Im(D''_\xi|_{\Omega^0}) = \hat{V}_\xi$, the quotient being $\hat{\theta}^H$, and

(2) Θ is holomorphic with respect to the holomorphic structure induced by $\hat{\mathrm{d}}^{0,1}$.

Recall from Section 2.3 that $Ker(D''_\xi|_{\Omega^1})$ is a closed Hilbert subspace of $\hat{H}_\xi$; call $\pi_{Ker(D''_\xi)}$ orthogonal projection of $\hat{H}_\xi$ to it. We now claim

that the map

$$\Theta : Ker(D''_\xi|_{\Omega^1}) \longrightarrow Ker(D''_\xi|_{\Omega^1}) \otimes \Omega^{1,0}_{\hat{\mathbf{C}}}$$

$$\tilde{f}_\xi \mapsto -\frac{1}{2}\pi_{Ker(D''_\xi)}(z\tilde{f}_\xi(z))\mathrm{d}\xi$$

verifies the hypotheses needed.

For (1), we need to show $\Theta(Im(D''_\xi|_{\Omega^0})) \subseteq Im(D''_\xi|_{\Omega^0})$. Let g_ξ be a local section of the trivial Hilbert bundle $L^2(E) \to \hat{\mathbf{C}}$. Then we have

$$\Theta(D''_\xi g) = -\frac{1}{2}\pi_{Ker(D''_\xi)}(zD''_\xi g_\xi)\mathrm{d}\xi$$

$$= -\frac{1}{2}\pi_{Ker(D''_\xi)}(D''_\xi(zg_\xi(z)))\mathrm{d}\xi$$

$$= -\frac{1}{2}D''_\xi(zg_\xi(z))\mathrm{d}\xi,$$

because the operator $D''_\xi = \bar\partial^{\mathcal{E}} + \theta_\xi$ commutes with multiplication by z, and $Im(D''_\xi|_{\Omega^0}) \subseteq Ker(D''_\xi|_{\Omega^1})$. This shows that $Im(D''_\xi|_{\Omega^0})$ is invariant by Θ; the quotient is clearly $\hat\theta^H$.

Next come to (2): we remark that the formula defining Θ only depends on ξ via the projection $\pi_{Ker(D''_\xi)}$. But since the operator D''_ξ depends holomorphically in ξ, so do the subspaces $Ker(D''_\xi)$, and since the metric is independent of ξ, the same thing is true for the projections $\pi_{Ker(D''_\xi)}$. This shows that Θ, and so $\hat\theta^H$ is holomorphic in ξ. $\qquad\square$

4.3. Identification with hypercohomology

In this section we will often use basic properties of hypercohomology; for an introduction to this topic, we refer to [**11**] and [**9**].

Before we start, we need to define the functional spaces

$$\tilde{L}^2_\xi(E) = \mathrm{Dom}_{\max}(D''_\xi|_{\Omega^0 \otimes E})$$

$$= \{u \in L^2(E) \ : \ \theta_\xi \wedge u, \bar\partial^{\mathcal{E}} u \in L^2\}$$

$$\tilde{L}^2_\xi(\Omega^{0,1} \otimes E) = \mathrm{Dom}_{\max}(D''_\xi|_{\Omega^{0,1} \otimes E})$$

$$= \{v\mathrm{d}\bar z \in L^2(\Omega^{0,1} \otimes E) \ : \ \theta_\xi \wedge v\mathrm{d}\bar z \in L^2\}$$

$$\tilde{L}^2(\Omega^{1,0} \otimes E) = \mathrm{Dom}_{\max}(D''_\xi|_{\Omega^{1,0} \otimes E})$$

$$= \{u\mathrm{d}z \in L^2(\Omega^{1,0} \otimes E) \ : \ \bar\partial^{\mathcal{E}}(u\mathrm{d}z) \in L^2\},$$

for the Euclidean metric $|\mathrm{d}z|^2$ on $\mathbf{C}$ and the hermitian metric h on the fibers, adapted to the parabolic structure with weights $\{\alpha^j_1, \ldots, \alpha^j_r\}$. Notice that we may drop the index ξ of these spaces, since they all coincide: indeed, in a logarithmic singularity the deformation $\xi\mathrm{d}z$ is

bounded, and at infinity the condition $\xi \notin \hat{P}$ implies that no eigenvalues of θ_ξ vanish, and this gives equivalence of the corresponding norms exactly as in Lemma 2.3. We identify these functional spaces to the sheaves of their local sections. In what follows, we are going to define sheaves $\mathcal{E}$ and $\mathcal{F}$ of sections of $\Omega^0 \otimes E$ and $\Omega^{1,0} \otimes E$ respectively on $\mathbf{CP}^1$ with the property that the L^2-cohomology $L^2 H^\bullet(D''_\xi)$ of (4.10) identifies to the hypercohomology $\mathbb{H}^\bullet(\mathcal{E} \xrightarrow{\theta_\xi \wedge} \mathcal{F})$ of the sheaf map $\mathcal{E} \xrightarrow{\theta_\xi \wedge} \mathcal{F}$. This latter is then explicitly given in terms of a sky-scraper sheaf over the zero set Σ_ξ of $det(\theta_\xi)$ by a simple use of the spectral sequence of the double complex.

4.3.1. Definition and resolution of the sheaves. Recall that the parabolic structure on $\mathcal{E}$ with adapted Hermitian fiber metric means that the holomorphic bundle $\mathcal{E}$ on $\mathbf{C} \setminus P$ has a natural extension to all $\mathbf{CP}^1$: the holomorphic sections at a singular point are the holomorphic sections outside the singularity which are bounded with respect to the metric. By an abuse of language, for $U \subset \mathbf{CP}^1$ an open set let $\mathcal{E}|_U$ be the set of holomorphic sections of the bundle $\mathcal{E}$ in U. In other words, we denote by $\mathcal{E}$ the sheaf of local holomorphic sections of $\mathcal{E}$ (extended over the punctures as above).

Next, let us define $\mathcal{F}$: for an open set $U \subset \mathbf{CP}^1$ containing no singular point, let $\mathcal{F}|_U$ be the set of $\bar{\partial}^\mathcal{E}$-holomorphic sections of $\Omega^{1,0} \otimes E$. If U contains exactly one singular point $p_j \in P$ (and does not contain the infinity), then let $\mathcal{F}|_U$ be the set of $\bar{\partial}^\mathcal{E}$-meromorphic sections σdz of $\Omega^{1,0} \otimes E$ such that σ be $\bar{\partial}^\mathcal{E}$-meromorphic in U with only one simple pole at p_j, and such that its residue in this point be contained in the subspace $Im(Res(\theta, p_j))$. Finally, if U contains the infinity (but no other singular points), then let $\mathcal{F}|_U$ be the set of all $\bar{\partial}^\mathcal{E}$-meromorphic sections σdz of $\Omega^{1,0} \otimes E$ with a double pole at infinity, and no other poles in U. Notice that since in the coordinate $w = 1/z$ of $\mathbf{CP}^1$ the section dz has a double pole at infinity, this amounts to say that σ is a $\bar{\partial}^\mathcal{E}$-holomorphic section of E in U. Writing $\sigma = \sum_k f_k^\infty \sigma_k^\infty$ in the holomorphic basis (1.30) at infinity, it is still the same thing to say that f_k^∞ be a holomorphic function in U for all k (in particular bounded at infinity). It is easy to check that this way we defined a sheaf.

We introduce some further notation: set $\tilde{r} = \sqrt{1 + |z|^2}$ on $\mathbf{C}$; then for $a \in \{0, 1\}$ we denote by $\tilde{r}\tilde{L}^2(\Omega^{a,0} \otimes E)$ the space of sections u of $\Omega^{a,0} \otimes E$ such that $\tilde{r}^{-1}u \in \tilde{L}^2$. This way we only loosen the condition

on the behavior of u at infinity with respect to $\tilde{L}^2$, namely that $r^{-1}u$ be in $\tilde{L}^2$ in a neighborhood of infinity. It is immediate that there exist an inclusion of vector spaces

$$(4.16) \qquad \tilde{L}^2(\Omega^{a,0} \otimes E) \hookrightarrow \tilde{r}\tilde{L}^2(\Omega^{a,0} \otimes E).$$

LEMMA 4.10. *The sequence*

$$(4.17) \qquad \mathcal{E} \hookrightarrow \tilde{r}\tilde{L}^2(E) \xrightarrow{\bar{\partial}^{\varepsilon}} \tilde{L}^2(\Omega^{0,1} \otimes E)$$

is a resolution of $\mathcal{E}$.

PROOF. It is known that away from the singularities, the sequence of usual L^2-sections with respect to Euclidean metric gives a resolution of the sheaf of holomorphic sections. Therefore, we only need to show that (4.17) is a resolution at the singularities.

Consider first $p_j \in P$. We first prove that (4.17) is locally exact in $\tilde{r}\tilde{L}^2(E)$. Let E be trivialized in $\Delta(p_j, \varepsilon)$ by the local sections $\{\sigma_k^j\}$ given in (1.27). As we have seen in (1.28), in this trivialisation up to a perturbation term $\theta = diag(\lambda_k^j)\mathrm{d}z/z$, with $\lambda_k^j = (\mu_k^j - \beta_k^j)/2$, and the parabolic weights are given by $\alpha_k^j = \Re(\mu_k^j) - [\Re(\mu_k^j)]$. By definition, any holomorphic section σ of E^j can be given as a sum $\sum_k \phi_k^j \sigma_k^j$, where ϕ_k^j are holomorphic functions defined in $\Delta(p_j, \varepsilon)$, in particular bounded by a constant K. This implies that $\sigma \in L^2(E)$, so that $\sigma \in \tilde{L}^2(E)$ if and only if $\theta \wedge \sigma \in L^2$. Recall that L^2 is defined with respect to the parabolic structure $\{\alpha_k^j\}$, and that the perturbation term in θ behaves as $O(r^{-1+\delta})$ with $\delta > 0$, where $r = |z - p_j|$. This implies that

$$\int_{\Delta(p_j,\varepsilon)} |\theta\sigma|^2 \leq K' \int \sum_{k=1}^{r_j} |r^{-1+\delta}\sigma_k^j|^2 + K' \int \sum_{k=r_j+1}^{r} |r^{-1}\sigma_k^j|^2$$

$$\leq K'' \int \sum_{k=1}^{r_j} |r^{-1+\delta}|^2 + K'' \int \sum_{k=r_j+1}^{r} |r^{-1+\alpha_k^j}|^2.$$

By Hypothesis 1.28, $\alpha_k^j > 0$ for all $j \in \{r_j + 1, \dots, r\}$. It then follows that this last expression is finite, which proves that any holomorphic section of E is in $\tilde{L}^2$. On the other hand, if a section $\sigma = \sum_k \phi_k^j \sigma_k^j$ of E is meromorphic in p_j, then there is at least one $k \in \{1, \dots, r\}$ such that ϕ_k^j has a pole in p_j. Suppose $k \in \{1, \dots, r_j\}$: then $|\phi_k^j \sigma_k^j| \sim 1/r$, and σ is clearly not in L^2. Suppose now $k \in \{r_j + 1, \dots, r\}$: then again by Hypothesis 1.28 we have $\lambda_k^j \neq 0$, and therefore $|\theta \wedge \phi_k^j \sigma_k^j| \sim r^{-2+\delta}$, and so $\theta \wedge \sigma \notin L^2$. Hence, the sections of $\tilde{L}^2(\Delta(p_j, \varepsilon), E)$ in the kernel

of $\bar{\partial}^{\mathcal{E}}$ are exactly the local holomorphic sections of E, in other words the local sections of $\mathcal{E}$. This shows local exactness in $\tilde{L}^2(E)$.

The next thing we show is that in $\Delta(p_j, \varepsilon)$ the complex (4.17) is exact at $\tilde{L}^2(\Omega^{0,1} \otimes E)$: let $v\mathrm{d}\bar{z} \in \tilde{L}^2(\Delta(p_j, \varepsilon), \Omega^{0,1} \otimes E)$ be an arbitrary section; for $\varepsilon > 0$ sufficiently small we wish to find $u \in \tilde{L}^2(\Delta(p_j, \varepsilon), E)$ such that

$$(4.18) \qquad\qquad \bar{\partial}^{\mathcal{E}} u = v\mathrm{d}\bar{z}$$

We can suppose without restricting generality that $v = f\sigma_k^j$, with f a function defined in $\Delta(p_j, \varepsilon)$. Since σ_k^j is a holomorphic section of E, solving (4.18) boils down to solving the usual Cauchy-Riemann equation on the disk

$$(4.19) \qquad\qquad \frac{\partial g}{\partial \bar{z}} = f$$

with $u = g\sigma_k^j \in \tilde{L}^2(\Delta(p_j, \varepsilon), E)$. Exactness near a singularity at a finite point is given by the following claim:

CLAIM 4.11. *For $f \in L^2$ the equation (4.19) has a solution g such that $gr^{-1+\delta} \in L^2$ for any $\delta > 0$. For f such that $fr^\alpha \in L^2$ with $0 < \alpha < 1$, (4.19) has a solution g such that $gr^{-1+\alpha} \in L^2$.*

PROOF. The first statement is established combining the usual resolution of the Cauchy-Riemann equation for $f \in L^2$ by an $L^{2,1}$-function g and the estimation (2.12).

The second one is a direct consequence of Proposition I.3 of [4]. One might also prove it by direct estimations on the solution given by the Cauchy kernel, as in [2]. $\qquad\square$

Now let us come back to exactness at a singularity in a finite point: for the regular case $k \in \{1, \ldots, r_j\}$ we have $f \in L^2$ and $|\theta \wedge g\sigma_k^j| \leq |g|r^{-1+\delta}$, so we can apply directly the first statement of the claim; for the singular case $k \in \{r_j+1, \ldots, r\}$ by definition $|\theta \wedge f\sigma_k^j \mathrm{d}\bar{z}| \sim |f|r^{-1+\alpha}$ is in L^2 with $\alpha > 0$ by Hypothesis 1.28, therefore we can apply the second statement of the claim. Remark that in this case even a stronger condition then the assumption $fr^\alpha \in L^2$ of the claim holds. However, we will need the claim in its full generality to show exactness at infinity.

We now come to exactness at infinity. Recall that $\xi \notin \hat{P}$ implies θ_ξ is an isomorphism $L^2(\Omega^{0,b}) \to L^2(\Omega^{1,b})$ for $b \in \{0, 1\}$. Therefore, the sections at infinity of the sheaves $\tilde{L}^2(\Omega^{0,b})$ and $L^2(\Omega^{0,b})$ coincide. First, we consider exactness in $\tilde{r}\tilde{L}^2(E) = \tilde{r}L^2(E)$: by the definition of $\mathcal{E}$, its

local sections are the holomorphic linear combinations $\sigma = \sum_k \phi_k^\infty \sigma_k^\infty$. First we check that these sections verify $r^{-1}\sigma \in L^2$: since $|\phi_k^\infty| \leq K$ and $|\sigma_k^\infty| \sim r^{-\alpha_k^\infty}$ with $\alpha_k^\infty > 0$ by Hypothesis 1.28, we see that $r^{-1}\phi_k^\infty \sigma_k^\infty \in L^2$. On the other hand, if we have a section $\sigma = \sum_k \phi_k^\infty \sigma_k^\infty$ in the kernel of $\bar{\partial}^\mathcal{E}$, then for all k the function ϕ_k^∞ is either holomorphic or meromorphic; but if $r^{-1}\sigma \in L^2$, then it implies that ϕ_k^∞ is holomorphic for all k. This proves exactness in the first term.

Next we come to the term $L^2(\Omega^{0,1} \otimes E)$: for a section $v\mathrm{d}\bar{z} \in L^2(\mathbf{C} \setminus \Delta(R), \Omega^{0,1} \otimes E)$ we search $u \in rL^2(\mathbf{C} \setminus \Delta(R), E)$ such that $\bar{\partial}^\mathcal{E} u = v$. Suppose $v = f\sigma_k^\infty$ and $u = g\sigma_k^\infty$ again. In the coordinate $w = 1/z = \rho e^{-\theta}$ on $\Delta(0, 1/R)$ we find (for simplicity we took $R = 1$ and wrote $\Delta = \Delta(0, 1/R)$):

$$\int_\Delta |f|^2 \rho^{2\alpha-4} |\mathrm{d}w|^2 = \int_{\mathbf{C} \setminus \Delta} |f|^2 r^{-2\alpha} |\mathrm{d}z|^2 < \infty$$

$$\int_\Delta |g|^2 \rho^{2\alpha-2} |\mathrm{d}w|^2 = \int_{\mathbf{C} \setminus \Delta} |g|^2 r^{-2-2\alpha} |\mathrm{d}z|^2 < \infty.$$

On the other hand, the Cauchy-Riemann equation

$$\frac{\partial g}{\partial \bar{z}} = f$$

transforms into

$$\frac{\partial g}{\partial \bar{w}} = -\frac{f}{\bar{w}^2}.$$

and we conclude applying Claim 4.11 to $-f/\bar{w}^2$.

$\square$

We can also show the counterpart of Lemma 4.10 for $\mathcal{F}$:

LEMMA 4.12. *The complex*

(4.20) $$\mathcal{F} \hookrightarrow \tilde{r}\tilde{L}^2(\Omega^{1,0} \otimes E) \xrightarrow{\bar{\partial}^\mathcal{E}} L^2(\Omega^{1,1} \otimes E)$$

is a resolution of $\mathcal{F}$.

PROOF. Away from the singularities this is also given by classical elliptic theory, therefore we focus our attention on a neighborhood of a singular point.

Let us first treat the case of a singularity at a finite point $p_j \in P$. A local section of $\mathcal{F}$ is then by definition a section $\sigma = \sum_k \phi_k^j \sigma_k^j \mathrm{d}z$ such that ϕ_k^j is holomorphic for $k \in \{1, \ldots r_j\}$ and has a pole of order at most one in p_j for $k \in \{r_j+1, \ldots r\}$. From the form of the parabolic structure, it follows that $|\phi_k^j \sigma_k^j| \sim O(1)$ for $k \in \{1, \ldots r_j\}$ and $|\phi_k^j \sigma_k^j| \sim O(r^{-1+\alpha_k^j})$

for $k \in \{r_j + 1, \ldots r\}$. By Hypothesis 1.28 we have $\alpha_k^j > 0$, thus $\sigma \in L^2(\Omega^{1,0} \otimes E)$. On the other hand, if a section $\sigma = \sum_k \phi_k^j \sigma_k^j \mathrm{d}z$ of $\Omega^{1,0} \otimes E$ satisfies $\bar{\partial}^{\mathcal{E}} \sigma = 0$, but $\sigma \notin L^2(\Omega^{1,0} \otimes E)$ then either ϕ_k^j has a pole for some $k \in \{1, \ldots r_j\}$ or ϕ_k^j has an at least double pole for some $k \in \{r_j + 1, \ldots r\}$, and therefore σ is not a local section of $\mathcal{F}$. This shows exactness in the first term.

Consider now exactness at the second term in $\Delta(p_j, \varepsilon)$: here we need to solve (4.19), for $f \in L^2$ with the solution g in L^2 in the regular case; and for f such that $fr^\alpha \in L^2$ with the solution g such that $gr^\alpha \in L^2$ in the singular case. Both follow from Claim 4.11.

There now remains to show exactness at infinity: this is done similarly to the case of $\mathcal{E}$. $\square$

4.3.2. Hypercohomology and L^2-cohomology. We can use the results of the last section in order to deduce the following:

PROPOSITION 4.13. The first L^2-cohomology $\hat{V}_\xi = L^2 H^1(D_\xi'')$ of (4.10) is isomorphic to the hypercohomology $\mathbb{H}^1(\mathcal{E} \xrightarrow{\theta_\xi \wedge} \mathcal{F})$.

PROOF. By Lemmas 4.10 and 4.12, θ_ξ defines a morphism of resolutions

$$(4.21) \qquad
\begin{array}{ccc}
\tilde{L}^2(\Omega^{0,1} \otimes E) & \xrightarrow{\theta_\xi \wedge} & L^2(\Omega^{1,1} \otimes E) \\
{\scriptstyle \bar{\partial}^{\mathcal{E}}} \uparrow & & \uparrow {\scriptstyle \bar{\partial}^{\mathcal{E}}} \\
\tilde{r}\tilde{L}^2(E) & \xrightarrow{\theta_\xi \wedge} & \tilde{r}\tilde{L}^2(\Omega^{1,0} \otimes E) \\
\cup \uparrow & & \uparrow \cup \\
\mathcal{E} & \xrightarrow{\theta_\xi \wedge} & \mathcal{F}
\end{array}$$

Therefore, by general theory, the hypercohomology of the sheaf map $\mathcal{E} \xrightarrow{\theta_\xi \wedge} \mathcal{F}$ identifies to the cohomology of the single complex formed by the double complex $\mathcal{D}_\xi^r$:

$$(4.22) \qquad
\begin{array}{ccc}
\tilde{L}^2(\Omega^{0,1} \otimes E) & \xrightarrow{\theta_\xi \wedge} & L^2(\Omega^{1,1} \otimes E) \\
{\scriptstyle \bar{\partial}^{\mathcal{E}}} \uparrow & & \uparrow {\scriptstyle \bar{\partial}^{\mathcal{E}}} \\
\tilde{r}\tilde{L}^2(E) & \xrightarrow{\theta_\xi \wedge} & \tilde{r}\tilde{L}^2(\Omega^{1,0} \otimes E).
\end{array}$$

We show that the first cohomology of the single complex of this double complex is isomorphic to the first cohomology of the single complex

associated to the double complex $\mathcal{D}_\xi$:

$$(4.23) \qquad \begin{array}{ccc} \tilde{L}^2(\Omega^{0,1} \otimes E) & \xrightarrow{\theta_\xi \wedge} & L^2(\Omega^{1,1} \otimes E) \\ {\scriptstyle \bar{\partial}^\mathcal{E}} \uparrow & & \uparrow {\scriptstyle \bar{\partial}^\mathcal{E}} \\ \tilde{L}^2(E) & \xrightarrow{\theta_\xi \wedge} & \tilde{L}^2(\Omega^{1,0} \otimes E). \end{array}$$

We define a map

$$\iota : H^1(\mathcal{D}_\xi) \longrightarrow H^1(\mathcal{D}_\xi^r)$$

as follows: represent a cohomology class of $H^1(\mathcal{D}_\xi)$ by a couple

$$(\kappa \mathrm{d}\bar{z}, \nu \mathrm{d}z) \in \tilde{L}^2(\Omega^{0,1} \otimes E) \oplus \tilde{L}^2(\Omega^{1,0} \otimes E),$$

and use the inclusion (4.16) to map it into the cohomology class represented by the same couple (κ, ν) in $H^1(\mathcal{D}_\xi^r)$. This is well defined, since if $(\kappa \mathrm{d}\bar{z} + \bar{\partial}^\mathcal{E}\lambda, \nu \mathrm{d}z + \theta_\xi \lambda)$ is a couple in $H^1(\mathcal{D}_\xi)$ representing the same class as $(\kappa \mathrm{d}\bar{z}, \nu \mathrm{d}z)$, for $\lambda \in \tilde{L}^2(E)$, then in particular $\lambda \in \tilde{r}\tilde{L}^2(E)$, and so the two couples are cohomologuous in $H^1(\mathcal{D}_\xi^r)$ as well. This also shows that ι is injective.

We only need to prove surjectivity: suppose we have a couple $(\kappa \mathrm{d}\bar{z}, \nu \mathrm{d}z) \in \tilde{L}^2(\Omega^{0,1} \otimes E) \oplus \tilde{r}\tilde{L}^2(\Omega^{1,0} \otimes E)$ representing a class in $H^1(\mathcal{D}_\xi^r)$. It is clearly sufficient to prove that this class can be represented by a couple vanishing in a neighborhood of infinity. Since θ_ξ is an isomorphism at infinity, we can put (restricting to a smaller neighborhood of infinity if necessary) $\lambda = \theta_\xi^{-1}(\nu \mathrm{d}z)$. This is then a section in $\tilde{r}\tilde{L}^2(E)$, and the couple $(\kappa \mathrm{d}\bar{z} - \bar{\partial}^\mathcal{E}\lambda, \nu \mathrm{d}z - \theta_\xi \lambda)$ is cohomologuous to $(\kappa \mathrm{d}\bar{z}, \nu \mathrm{d}z)$ in $H^1(\mathcal{D}_\xi^r)$. By definition, the $(1,0)$-term of this couple vanishes at infinity. The same thing is true for the $(0,1)$-part, because $\theta_\xi(\kappa \mathrm{d}\bar{z} - \bar{\partial}^\mathcal{E}\lambda) = -\bar{\partial}^\mathcal{E}(\nu \mathrm{d}z - \theta_\xi \lambda) = 0$ near infinity and θ_ξ is an isomorphism there. This finishes the proof of the proposition, for the L^2-cohomology of (4.10) is by definition the cohomology of the single complex associated to $\mathcal{D}_\xi$. □

4.3.3. The spectral curve. In the explicit identification of the hypercohomology, the following notions will be of much importance. Recall that (up to wedge product by $\mathrm{d}z$) θ_ξ is a meromorphic section of $End(E)$ over $\mathbf{CP}^1$.

DEFINITION 4.14. *For $\xi \in \hat{\mathbf{C}} \setminus \hat{P}$, the set of zeros of $det(\theta_\xi)$ is called the* spectral set *corresponding to ξ. We denote it by Σ_ξ.*

LEMMA 4.15. *For each $\xi \in \hat{\mathbf{C}} \smallsetminus \hat{P}$, the spectral set is an effective divisor of $\mathbf{CP}^1$, in other words a finite set of points with multiplicities in $\mathbf{N}$.*

PROOF. The section $\det(\theta_\xi)$ of $End(V)$ is holomorphic with respect to $\bar{\partial}^{\mathcal{E}}$. We only need to check it does not vanish identically for any ξ. Suppose there exists ξ such that

$$det(\theta_\xi(q)) = 0$$

for all $q \in \mathbf{C} \smallsetminus P$. In different terms, θ has a constant eigenvalue over $\mathbf{C} \smallsetminus P$; in particular, the residue of this eigenvalue at infinity is 0. This contradicts $\lambda_k^\infty \neq 0$ for all $k \in \{1, \ldots, n\}$ (see (2) of Hypothesis 1.28). □

A basic property is the following.

CLAIM 4.16. *The points of Σ_ξ define a multi-valued meromorphic function of $\xi \in \hat{\mathbf{C}}$.*

PROOF. By assumption, $det(\theta_\xi(z))$ depends holomorphically on $\xi \in \hat{\mathbf{C}}$ and meromorphically on z. We conclude using the implicit function theorem, namely that the solutions of a meromorphic equation depending holomorphically on a variable are meromorphic in this variable. □

DEFINITION 4.17. *The graph of the multi-valued meromorphic function*

$$\hat{\mathbf{C}} \smallsetminus \hat{P} \longrightarrow \mathbf{CP}^1$$

$$\xi \mapsto \Sigma_\xi$$

is called the spectral curve *of the Higgs bundle. It is denoted by Σ.*

This object was first studied by N. Hitchin in [**13**]. By Claim 4.16 the spectral curve is an analytic subvariety

$$\Sigma \xrightarrow{\jmath} (\hat{\mathbf{C}} \smallsetminus \hat{P}) \times \mathbf{CP}^1,$$

of (complex) dimension one. (Here $\jmath$ stands for inclusion.) Moreover, by construction it is naturally a branched cover of $\hat{\mathbf{C}}$ via projection to the first factor.

Here is an important property.

PROPOSITION 4.18. *The spectral curve Σ is reduced; in other words, $det(\theta_\xi)$ vanishes only up to the first order except for a finite set of points of Σ.*

PROOF. Suppose Σ has infinitely many points (q, ξ) where $det(\theta_\xi)$ vanishes up to order higher than one. Since Σ has a natural extension into a compact curve in $\mathbf{CP}^1 \times \widehat{\mathbf{CP}}^1$ (see Section 4.4), this means that for any ξ some zero $q(\xi) \in \Sigma_\xi$ of θ_ξ has multiplicity higher than one; in different terms, some irreducible component of Σ has multiplicity higher than one. In particular, as $\xi \to \infty$, at least two of the $q_k(\xi)$ must have the same Laurent expansions. This is impossible by (4.37) and the assumption $\lambda_k^j \neq \lambda_{k'}^j$ for $k \neq k'$ made in (1) of Hypothesis 1.28. $\qquad\square$

4.3.4. Explicit computation of the hypercohomology. Let us now compute the hypercohomology of

$$(4.24) \qquad\qquad \mathcal{E} \xrightarrow{\theta_\xi \wedge} \mathcal{F}$$

Consider arbitrary algebraic resolutions of the sheaves $\mathcal{E}$ and $\mathcal{F}$ such that $\theta_\xi \wedge$ induce a morphism of resolutions

$$(4.25)$$

$$
\begin{array}{ccc}
K^{0,1} & \xrightarrow{\theta_\xi \wedge} & K^{1,1} \\
{\scriptstyle\delta}\big\uparrow & & \big\uparrow{\scriptstyle\delta} \\
K^{0,0} & \xrightarrow{\theta_\xi \wedge} & K^{1,0} \\
\big\uparrow & & \big\downarrow \\
\mathcal{E} & \xrightarrow{\theta_\xi \wedge} & \mathcal{F}.
\end{array}
$$

For example, one might take resolutions by Čech cochains. By definition, the first filtration K_p of the single complex associated to (4.25) is given by

$$K_0 = (K^{0,1} \oplus K^{0,0}) \oplus (K^{1,1} \oplus K^{1,0})$$
$$K_1 = K^{1,1} \oplus K^{1,0}.$$

The first page of the spectral sequence corresponding to this filtration is given by

$$(4.26)$$

$$
\begin{array}{ccc}
(\mathcal{H}^0)^{[1]}(\mathbf{CP}^1) & & (\mathcal{H}^1)^{[1]}(\mathbf{CP}^1) \\
{\scriptstyle\delta}\big\uparrow & & \big\uparrow{\scriptstyle\delta} \\
(\mathcal{H}^0)^{[0]}(\mathbf{CP}^1) & & (\mathcal{H}^1)^{[0]}(\mathbf{CP}^1)
\end{array}
$$

where $\mathcal{H}^j$ is the j-th cohomology sheaf of the map (4.24), and the vertical sequences come from resolutions

$$\mathcal{H}^0 \hookrightarrow (\mathcal{H}^0)^{[0]} \xrightarrow{\delta} (\mathcal{H}^0)^{[1]}$$

$$\mathcal{H}^1 \hookrightarrow (\mathcal{H}^1)^{[0]} \xrightarrow{\delta} (\mathcal{H}^1)^{[1]}$$

by taking global sections. Let us now describe explicitly the cohomology sheaves. Recall from definition 4.14 that $q \in \Sigma_\xi$ are exactly the points where the map $\theta_\xi(q) : E(q) \to E(q)$ is not surjective. After all this preparation, we have the following characterization:

LEMMA 4.19. *The cohomology sheaf $\mathcal{H}^0$ of order 0 of the sheaf map (4.24) is 0. If $\det(\theta_\xi)$ has a zero of order 1 in all points of $q \in \Sigma_\xi$, then the first cohomology sheaf $\mathcal{H}^1$ is the sky-scraper sheaf $\mathcal{R}_\xi$ whose stalk over a point $q \in \Sigma_\xi$ is the finite-dimensional subspace $coKer(\theta_\xi(q)) \subset E(q)$, and all other stalks are 0.*

REMARK 4.20. *The cokernel of $\theta_\xi(q)$ is naturally identified with the orthogonal of the image with respect to the fiber metric, or, which is the same thing, with the kernel of $\theta_\xi^*(q)$. This allows us to think of $coKer(\theta_\xi(q))$ as a subspace of $E(q)$.*

PROOF. Let us start with $\mathcal{H}^0$: suppose we have a section $\phi \in \mathcal{E}|_U$ on an open set $U \subset \mathbf{CP}^1$ such that $\theta_\xi \phi = 0$. Since on the open subset $U \smallsetminus \Sigma_\xi$ the map $\theta_\xi : E(q) \to E(q)$ is an isomorphism, we deduce that $\phi = 0$ on this set. But a holomorphic section vanishing on an open set vanishes everywhere, thus $\phi = 0$ on all of U. This gives the first statement of the lemma.

We now come to $\mathcal{H}^1$: let $U \subset \mathbf{CP}^1$ be an open subset. If $U \cap \Sigma_\xi = \emptyset$ then θ_ξ is an invertible holomorphic endomorphism of $\mathcal{E}$ on U, therefore $\mathcal{H}^1|_U = 0$. Suppose now U contains exactly one point $q \in \Sigma_\xi$. Then, for any section $\phi \in \mathcal{E}|_U$ the vector $(\theta_\xi \phi)(q)$ lies by definition in the image of $\theta_\xi(q)$, which is just the orthogonal of $coKer(\theta_\xi(q))$. Therefore, this latter is contained in $\mathcal{H}^1|_U$. On the other hand, the condition that θ_ξ has a zero of order 1 in q means that any section $\psi \in \mathcal{E}|_U$ such that $\psi(q) \perp coKer(\theta_\xi(q))$ is in $Im(\theta_\xi)$. This proves the second statement. $\square$

REMARK 4.21. *By Proposition 4.18, the condition of $\det(\theta_\xi)$ having a first-order zero in all points of Σ_ξ is generic in ξ: it is verified for all ξ except for twice the eigenvalues of $\theta(q)$ for the finite number of points*

q of Σ of multiplicity higher than one. For the discrete set of ξ where there exists a $q \in \Sigma_\xi$ with a multiple zero, one introduces the flag

$$E(q) = F_0 E(q) \supseteq coKer(\theta_\xi(q)) = F_1 E(q) \supseteq \ldots \supset F_{r_q} E(q) = \{0\},$$

the subscript of F being the order of zero of $\theta_\xi^*(q)$ along the given subspace, and proves that the cohomology sheaf $\mathcal{H}^1|_U$ over an open set containing q as the only element of Σ_ξ is in this case equal to the jet space

$$\bigoplus_{m=1}^{r_q-1} F_m E(q).$$

The assumptions that for fixed $j \in \{1, \ldots, n\}$ all the λ_k^j be different for $k \in \{r_j + 1, \ldots, r\}$ and for fixed $l \in \{1, \ldots, n'\}$ all the λ_k^∞ be different for $k \in \{1 + a_l, \ldots a_{l+1}\}$ (see (1) and (2), Hypothesis 1.28), mean that in the punctures of $\widehat{\mathbf{CP}}^1$ the limit states have first-order zeros.

Now since a resolution of the sky-scraper sheaf $\mathcal{R}_\xi$ is given by

$$\mathcal{R}_\xi \hookrightarrow \mathcal{R}_\xi \to 0,$$

the first page of the hypercohomology spectral sequence (4.26) becomes

$$
\begin{array}{ccc}
0 & & 0 \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\
0 & & \bigoplus_{q\in\Sigma_\xi} coKer(\theta_\xi(q)).
\end{array}
$$

All this implies the following:

PROPOSITION 4.22. *The hypercohomology spectral sequence corresponding to the first filtration collapses in its first page, and we have a natural isomorphism*

$$\mathbb{H}^1(\mathcal{E} \xrightarrow{\theta_\xi \wedge} \mathcal{F}) \simeq \bigoplus_{q\in\Sigma_\xi} coKer(\theta_\xi(q)).$$

PROOF. This is a consequence of the standard fact that a spectral sequence collapses as soon as non-zero elements only appear in one of its rows. Furthermore, an explicit isomorphism can be given as follows: fix a radially invariant bump-function χ on the unit disk $\Delta \subset \mathbf{C}$, equal to 0 on the boundary of Δ and to 1 in 0, and such that $d\chi$ is supported on the annulus $1/3 < r < 2/3$. For any complex number $a \neq 0$ set $\chi_a(z) = \chi(z/a)$. Now choose $\varepsilon_0 > 0$ so that the distance in $\mathbf{C}$ between any two distinct points of the finite set $P \cup \Sigma_\xi$ is at least $3\varepsilon_0$. For any

$(v_q)_{q\in\Sigma_\xi} \in \oplus coKer(\theta_\xi(q))$ consider the section $v_{\varepsilon_0} = \sum_{q\in\Sigma_\xi} v_q\chi_\varepsilon(z-q)$. Because $\mathrm{d}\chi_{\varepsilon_0}$ is supported on the annulus $\varepsilon_0/3 < r < 2\varepsilon_0/3$, the section $\bar{\partial}^\mathcal{E}(v_{\varepsilon_0}\mathrm{d}z) \in \Omega^{1,1}\otimes E$ is supported outside a neighborhood of Σ_ξ. Since this latter is the zero set of $\det(\theta_\xi)$, it then follows that there exists a section $t_{\varepsilon_0}\mathrm{d}\bar{z} \in \Omega^{0,1}\otimes E$ such that $\theta_\xi \wedge (t_{\varepsilon_0}\mathrm{d}\bar{z}) + \bar{\partial}^\mathcal{E}(v_{\varepsilon_0}\mathrm{d}z) = 0$, and t_{ε_0} is supported on the support of $\bar{\partial}^\mathcal{E}v_{\varepsilon_0}$, that is outside a neighborhood of Σ_ξ and of infinity. The couple $(v_{\varepsilon_0}\mathrm{d}z, t_{\varepsilon_0}\mathrm{d}\bar{z})$ therefore defines a cocycle in the single complex associated to $\mathcal{D}_\xi$, and using Proposition 4.13 we can define a map

$$\Psi_\xi : \bigoplus_{q\in\Sigma_\xi} coKer(\theta_\xi(q)) \longrightarrow H^1(\mathcal{D}_\xi) = \mathbb{H}^1(\mathcal{E} \xrightarrow{\theta_\xi\wedge} \mathcal{F})$$

(4.27) $$(v_q)_{q\in\Sigma_\xi} \mapsto [(v_{\varepsilon_0}\mathrm{d}z, t_{\varepsilon_0}\mathrm{d}\bar{z})],$$

where $[(v_{\varepsilon_0}\mathrm{d}z, t_{\varepsilon_0}\mathrm{d}\bar{z})]$ stands for the cohomology class in $H^1(\mathcal{D}_\xi)$ of this couple.

We need to show that this map does not depend on $\varepsilon_0 > 0$ chosen, provided that it is sufficiently small as explained above. Consider therefore the section v_{ε_1} for $\varepsilon_1 < \varepsilon$. Since in the union of the disks of radius $\varepsilon_1/3$ around the elements of Σ_ξ we have $v_{\varepsilon_1} = v_{\varepsilon_0}$, and θ_ξ is invertible outside this set, there exists a section $u \in \Gamma(E)$ such that $\theta_\xi u + v_{\varepsilon_1}\mathrm{d}z = v_{\varepsilon_0}\mathrm{d}z$. Then, as in the proof of Proposition 4.13, the couple $(v_{\varepsilon_0}\mathrm{d}z, t_{\varepsilon_0}\mathrm{d}\bar{z})$ is equal to $(v_{\varepsilon_1}\mathrm{d}z + \theta_\xi u, t_{\varepsilon_1}\mathrm{d}\bar{z} + \bar{\partial}^\mathcal{E}u)$, and the two couples define the same cohomology class in $H^1(\mathcal{D})$. This then allows us to fix $\varepsilon_0 > 0$ sufficiently small once and for all.

In a similar way, one can prove that Ψ_ξ is independent of the actual cut-off function χ as well.

Finally, the inverse of Ψ_ξ can be obtained as follows: let the cohomology class $\eta \in H^1(\mathcal{D}_\xi)$ be represented by a 1-form $\eta^{1,0}\mathrm{d}z + \eta^{0,1}\mathrm{d}\bar{z}$, where $\eta^{1,0}$ and $\eta^{0,1}$ are sections of E. Then we have

(4.28) $$\Psi_\xi^{-1}\eta = (eval_q\eta^{1,0})_{q\in\Sigma_\xi},$$

where $eval_q\eta^{1,0}$ stands for evaluation of the section $\eta^{1,0}$ in the point q. $\qquad\square$

REMARK 4.23. *Notice that the formula (4.28) is independent of the 1-form representative of η; in particular, the $(1,0)$-part of the harmonic representative of a cohomology class $\Psi_\xi(v_q)_{q\in\Sigma_\xi}$ vanishes in the $q \in \Sigma_\xi$ where $v_q = 0$.*

4.4. Extension of the Higgs bundle over the singularities

The interpretation of the holomorphic bundle underlying the transformed Higgs bundle in terms of hypercohomology established in the previous section allows us to extend it over the singular points $\hat{P} \cup \{\infty\}$ in the parameter space $\widehat{\mathbf{CP}}^1$. At each puncture, we need to do two things: first, define the fiber of the transformed vector bundle over it. This then extends the holomorphic structure induced by $\bar{\partial}^{\hat{\mathcal{E}}}$ over the puncture in a natural way: a holomorphic section through the singular point will be a continuous section in a neighborhood of it, that is holomorphic in the punctured neighborhood. (Continuity is defined at the same time as the exceptional fiber.) The second thing to do then is to give an explicit basis of holomorphic sections with respect to this extended holomorphic structure. It is important to note that the extensions $^i\hat{\mathcal{E}}$ we define here are *not* the transformed extensions given in Definition 3.11, but rather ones induced by the original Higgs bundle, and for which computations are more comfortable. This is why we will call $^i\hat{\mathcal{E}}$ the *induced extension*. We study the link between these two extensions in Section 4.7.

4.4.1. Extension to logarithmic singularities. First, we consider the case of points of the set $\hat{P}$. We shall now describe the extension $^i\hat{\mathcal{E}}$ over such a point. Notice first that as the deformation θ_ξ has a well-defined extension over these points, its hypercohomology spaces are also well-defined there. In particular, in view of Proposition 4.13, we may extend the transformed vector bundle $\hat{V}$ by putting

$$\hat{V}_{\xi_l} = \mathbb{H}^1(\mathcal{E} \xrightarrow{\theta_{\xi_l} \wedge} \mathcal{F})$$

This is the definition of the fiber over such a point.

In order to give explicit representatives of holomorphic sections, let us examine what happens to the fiber $\hat{V}_\xi$ when ξ approaches one of the points of $\hat{P} = \{\xi_1, \ldots, \xi_{n'}\}$, say ξ_l. First, let us find the spectral points.

CLAIM 4.24. *As $\xi \to \xi_l$, exactly $m_l = a_{l+1} - a_l$ branches of the meromorphic functions $q_k \in \Sigma_\xi$ converge to infinity, while all others remain in a bounded region of $\mathbf{C}$. Moreover, labelling the spectral points converging to infinity by $q_{1+a_l}(\xi), \ldots, q_{a_{l+1}}(\xi)$, they admit the asymptotic behavior*

$$(4.29) \qquad q_k(\xi) = \frac{2\lambda_k^\infty}{(\xi - \xi_l)} + O(|\xi - \xi_l|^{-\delta}),$$

where $\delta > 0$ can be chosen arbitrarily small. In particular, the branches converging to $\infty \in \mathbf{CP}^1$ of the spectral curve are not ramified over the point ξ_l.

PROOF. As it can be seen from (1.31), exactly m_l of the eigenvalues of the leading order term near infinity of the Higgs field θ_ξ converges to 0. Recall from Definition 4.14 that Σ_ξ is the vanishing set of $det(\theta_\xi)$. This implies that (counted with multiplicities) exactly m_l of the points $q(\xi) \in \Sigma_\xi$ converge to infinity; label these by $1 + a_l, \ldots, a_{l+1}$. All the other spectral points remain therefore bounded. By assumption (see (1.31)) in a holomorphic trivialisation of the bundle $\mathcal{E}$ in a neighborhood of $\infty \in \mathbf{CP}^1$, ignoring the factor dz the field θ_ξ is of the form

$$\frac{1}{2}(A - \xi \mathrm{Id}) + \frac{C}{z} + O(z^{-2}),$$

where $O(z^{-2})$ stands for holomorphic terms independent of ξ. Suppose first that the field is exactly equal to the polar part in this formula, in other words the $O(z^{-2})$ term is equal to 0. Then the solutions $\tilde{q}_1(\xi), \ldots, \tilde{q}_r(\xi)$ are clearly given by

$$\tilde{q}_k(\xi) = \frac{2\lambda_k^\infty}{(\xi - \xi_l)}.$$

In general, since $det(\theta_\xi)$ is holomorphic in z, we can apply Rouché's theorem to compare the position of the zeros of $det(\theta_\xi)$ with those of the polar part studied above. This yields that the solutions $q_k(\xi) \in \mathbf{C}$ of $det(\theta_\xi)(q(\xi)) = 0$ near infinity are close to $\tilde{q}_k(\xi)$; more precisely for any $\delta > 0$, there exists $K > 0$ such that for all $|\xi - \xi_l|$ sufficiently small we have

$$|q_k(\xi) - \tilde{q}_k(\xi)| < K|\xi - \xi_l|^{-\delta}.$$

Remark here that as $\xi \to \xi_l$ the behavior of $|\xi - \xi_l|^{-\delta}$ is small compared to $|\tilde{q}_k(\xi)| = c|\xi - \xi_l|^{-1}$. In other words, we have the expansion (4.29) so that $q_k(\xi)$ converges indeed to infinity asymptotically proportionally to $(\xi - \xi_l)^{-1}$ for $a_l < k \leq a_{l+1}$, while all other holomorphic families of zeros of $det(\theta_\xi)$ remain bounded.

The condition that the $\lambda_{1+a_l}, \ldots, \lambda_{a_{l+1}}$ are all distinct (see (2), Hypothesis 1.28) now implies that there is no splitting of the solutions at infinity, that is to say locally near $\xi = \xi_l$ any $q_k(\xi)$ with $a_l < k \leq a_{l+1}$ itself forms a meromorphic function without branching. Indeed, the occurrence of a branching at infinity implies that the Puiseux series of

the corresponding solutions agree, which is not the case here because of the asymptotic behaviors (4.29) with different leading coefficients. $\square$

Now, recall that for fixed $\xi \in \hat{\mathbf{C}} \smallsetminus \hat{P}$, in the explicit description of $\hat{V}_\xi$ given in the proof of Proposition 4.22, we considered the zeros $q_k(\xi)$ for $k = 1, \ldots, r$ of $det(\theta_\xi)(q)$, and for each $q_k(\xi)$ an element $v_k(\xi)$ of the subspace $coKer(\theta_\xi)_{q_k(\xi)} \subset E_{q_k(\xi)}$. Then we extended each $v_k(\xi)$ holomorphically into a neighborhood of $q_k(\xi)$, and multiplied the section we obtained by a bump-function equal to 1 in a small disk around $q_k(\xi)$ and to 0 on the boundary of a slightly larger disk. This section of $\mathcal{F}$ constituted the $(1, 0)$-part of the element in $\mathbb{H}^1(\mathcal{E} \xrightarrow{\theta_\xi} \mathcal{F}) \simeq \hat{V}_\xi$, and we chose the $(0, 1)$-part in such a way that the couple be in $Ker(D''_\xi)$. In what follows, we wish to do the same thing, but for all ξ in a neighborhood of ξ_l at the same time.

Let us consider one meromorphic family of zeros $q_k(\xi)$ with $a_l < k \le a_{l+1}$. We have just seen that $q_k(\xi)$ converges to ∞ as $\xi \to \xi_l$; therefore, we need to take a holomorphic section of $\mathcal{E}$ at infinity, extending an element of the cokernel of θ_{ξ_l}. One can check from formula (1.31) that this cokernel is equal to the vector subspace of the fiber $\mathcal{F}_\infty = E_\infty \otimes dz$ generated by $\{\sigma_m^\infty(\infty)dz\}_{m=1+a_l}^{a_{l+1}}$, where $\{\sigma_m^\infty\}_{m=1}^r$ is the holomorphic trivialisation of $\mathcal{E}$ at infinity considered in (1.30). Furthermore, since the metric h is mutually bounded with the diagonal model

$$diag(|z|^{-2\alpha_k^\infty}),$$

the orthogonal of the image of θ_ξ in $E(q_k(\xi))$ converges to $\sigma_k^\infty(\infty)$ as $\xi \to \xi_l$. Let $\varsigma_k(z)$ be a holomorphic extension of $\sigma_k^\infty(\infty)$ to a neighborhood of infinity such that for any $\xi \in \hat{\mathbf{C}}$ sufficiently close to ξ_l, the vector $\varsigma_k(q_k(\xi))dz$ be in the cokernel of $\theta_\xi(q_k(\xi))$. Such an extension exists because θ_ξ varies holomorphically with ξ and by Claim 4.24 $q_k(\xi)$ is a genuine (single-valued) meromorphic function of ξ. A holomorphic section $\hat{\sigma}_k$ of $\hat{\mathcal{E}}$ around ξ_l is then given by the section constructed as follows: for ξ sufficiently close to ξ_l such that ς_k is defined in $q_k(\xi)$, set

$$(4.30) \qquad v_k(z, \xi) = \chi_{\varepsilon_0(\xi-\xi_l)^{-1}}(z - q_k(\xi))\varsigma_k(z),$$

where we recall from the proof of Proposition 4.22 that $\chi_{\varepsilon_0(\xi-\xi_l)^{-1}}$ is a bump-function on a disk centered at 0 and of diameter $\varepsilon_0|\xi-\xi_l|^{-1}$ with ε_0 sufficiently small only depending on the parameters of the initial connection, fixed once and for all. (The importance of this choice will

become clear in Theorem 4.35.) Also, let $t_k(z,\xi)\mathrm{d}\bar{z} \in \Gamma(\mathbf{C}, E \otimes \Omega^{0,1})$ be the unique solution of the equation

$$(4.31) \qquad\qquad \bar{\partial}^{\mathcal{E}} v_k(z,\xi)\mathrm{d}z = -\theta_\xi t_k(z,\xi)\mathrm{d}\bar{z}.$$

Then consider the cohomology class $\hat{\sigma}_k^l(\xi)$ in $\mathbb{H}^1(\mathcal{E} \xrightarrow{\theta_\xi \wedge} \mathcal{F}) \simeq \hat{V}_\xi$ of the couple $(v_k(z,\xi)\mathrm{d}z, t_k(z,\xi)\mathrm{d}\bar{z})$ defined as above. Since the choice of ς_k is independent of ξ and moreover θ_ξ and $q_k(\xi)$ depend holomorphically on ξ, it follows that $\hat{\sigma}_k^l$ is $\bar{\partial}^{\hat{\mathcal{E}}}$-holomorphic in ξ outside of ξ_l.

DEFINITION 4.25. *Let the extension $^i\hat{\mathcal{E}}$ of $\hat{\mathcal{E}}$ to ξ_l be defined by the holomorphic trivialisation given by the sections $\hat{\sigma}_k^l$ for all choice of $k \in \{1+a_l, \ldots, a_{l+1}\}$ and for some holomorphic extension ς_k of $\sigma_k^\infty(\infty)$ such that for any $\xi \in \hat{\mathbf{C}}$ sufficiently close to ξ_l, we have $\varsigma_k(q_k(\xi))dz \in coKer(\theta_\xi(q_k(\xi)))$.*

4.4.2. Extension to infinity. In order to define the fiber over infinity, we first rephrase what we have done until now to obtain the holomorphic bundle $\hat{\mathcal{E}} = (\hat{V}, \bar{\partial}^{\hat{\mathcal{E}}})$ underlying the transformed Higgs bundle: we considered the sheaves $\mathcal{E}$ and $\mathcal{F}$ over $\mathbf{CP}^1$, we pulled them back to $\mathbf{CP}^1 \times \hat{\mathbf{C}}$ by the projection map π_1 on the first factor, and formed the sheaf map

$$\pi_1^* \mathcal{E} \xrightarrow{\theta_\bullet} \pi_1^* \mathcal{F}$$

equal to θ_ξ on the fiber $\mathbf{CP}^1 \times \{\xi\}$. We then defined the vector bundle

$$\hat{V}_\bullet = \mathbb{H}^1(\pi_1^* \mathcal{E} \xrightarrow{\theta_\bullet} \pi_1^* \mathcal{F}),$$

over $\hat{\mathbf{C}} \smallsetminus \hat{P}$ and we let $\bar{\partial}^{\hat{\mathcal{E}}}$ be the partial connection induced by $\hat{\mathrm{d}}^{0,1}$. In what follows, we keep on writing $\mathcal{E}$ and $\mathcal{F}$ for their pull-back to the product, whenever this does not cause confusion. Notice that $\theta_\bullet$ is holomorphic in both coordinates. We wish to extend the hypercohomology of this sheaf map over infinity; we will be done if we can extend the map $\theta_\bullet$ over infinity in a holomorphic manner. Indeed, the hypercohomology of a holomorphic family of sheaf morphisms is a holomorphic vector bundle over the base space of the deformations, in our case $\widehat{\mathbf{CP}}^1$. Notice that by definition $\theta_\xi = \theta - \xi/2\mathrm{d}z\wedge$, so it becomes singular as we let ξ converge to infinity. However, we can slightly change the sheaf $\mathcal{F}$ in such a way that there exist a natural extension of $\theta_\bullet$. Again, we follow [15].

Consider the projections π_j to the j-th coordinate in the product manifold $\mathbf{CP}^1 \times \widehat{\mathbf{CP}}^1$, and set $\tilde{\mathcal{F}} = \pi_2^* \mathcal{O}_{\widehat{\mathbf{CP}}^1}(1) \otimes \mathcal{F}$. Recall that $\mathcal{O}_{\widehat{\mathbf{CP}}^1}(1)$

admits two global holomorphic sections s_0 and s_∞, characterized by the fact that if $\hat{U}_0$ and $\hat{U}_\infty$ are the standard neighborhoods of $0 \in \widehat{\mathbf{CP}}^1$ and $\infty \in \widehat{\mathbf{CP}}^1$ with coordinates ξ and $\zeta = \xi^{-1}$ vanishing in 0 and ∞ respectively, then we have

$$(4.32) \qquad s_0(\xi) = \xi \qquad\qquad s_\infty(\xi) = 1 \qquad\qquad \text{in } \hat{U}_0$$

$$(4.33) \qquad s_0(\zeta) = 1 \qquad\qquad s_\infty(\zeta) = \zeta \qquad\qquad \text{in } \hat{U}_\infty.$$

Notice that here ξ is the standard coordinate of $\mathbf{C}$ we used to define θ_ξ. Therefore for $\eta \in \widehat{\mathbf{CP}}^1$ we put

$$(4.34) \qquad\qquad \tilde{\theta}_\eta : \mathcal{E} \longrightarrow \mathcal{F}$$

$$(4.35) \qquad\qquad \tilde{\theta}_\eta = s_\infty(\eta) \otimes \theta - \frac{1}{2} s_0(\eta) \otimes \mathrm{d}z\wedge,$$

We remark that by (4.32), on $\hat{U}_0 = \mathbf{C}$ we have $\tilde{\theta}_\xi = \theta - \xi/2\mathrm{d}z\wedge = \theta_\xi$, so $\tilde{\theta}_\bullet$ is indeed an extension of the deformation $\theta_\bullet$ to infinity. Therefore, in what follows we keep on writing θ for $\tilde{\theta}$ whenever this does not cause any confusion. In the same manner, we see that

$$\theta_\infty = -\frac{1}{2} s_0(\xi)_{\xi=\infty} \otimes \mathrm{d}z\wedge : \mathcal{E} \longrightarrow \mathcal{F} \otimes \mathcal{O}_{\widehat{\mathbf{CP}}^1}(1)_{\xi=\infty}.$$

From the definition of the sheaves $\mathcal{E}$ and $\mathcal{F}$ one can see that the co-homology sheaves of this map are $\mathcal{H}^0(\mathrm{d}z\wedge) = 0$ and $\mathcal{H}^1(\mathrm{d}z\wedge) = \mathcal{R}_\infty$, the sky-scraper sheaf supported in points of P and having stalk equal to $s_0(\xi)_{\xi=\infty} \otimes coKer(Res(\theta, p))$ in $p \in P$. Therefore, as in Proposition 4.22, we obtain that the first hypercohomology space of this map equals $s_0(\xi)_{\xi=\infty} \otimes (\oplus_{p\in P} coKer(Res(\theta, p)))$, and all its other hypercohomology spaces vanish. The extension of the vector bundle $\hat{V}$ to infinity is then given by setting $\hat{V}_\eta = \mathbb{H}^1(\mathcal{E} \xrightarrow{\theta_\eta} \mathcal{F})$ for all $\eta \in \widehat{\mathbf{CP}}^1 \smallsetminus \hat{P}$. In particular, any local section at $\zeta = 0$ of $\hat{E}$ is a family of sections of the sheaf $\tilde{\mathcal{F}}$, and therefore can be written

$$(4.36) \qquad\qquad s_0(\zeta) \otimes \psi(z, \zeta),$$

where $\psi(z, \zeta)$ are sections of $\mathcal{F}$ depending on the parameter ζ.

DEFINITION 4.26. *The extension $^i\hat{\mathcal{E}}$ of the holomorphic structure of $\hat{\mathcal{E}}$ to infinity is the extension whose holomorphic sections at infinity can be written as in (4.36), with $\psi(z, \zeta)$ holomorphic in ζ.*

We come to the explicit description of a holomorphic section of $^i\hat{\mathcal{E}}$ at $\xi = \infty$ with respect to this extension. We make a similar construction as in the case of logarithmic singularities: first, we make a basic remark.

CLAIM 4.27. *As $\xi \to \infty$, all zeros of $det(\theta_\xi)$ converge to one of the points of P. Moreover, supposing $q(\xi) \to p_j$, we have the asymptotic behavior*

$$(4.37) \qquad q(\xi) = p_j + 2\frac{\lambda_k^j}{\xi} + O(\xi^{-2+\delta}),$$

where λ_k^j is a non-vanishing eigenvalue of the residue of θ at p_j and $\delta > 0$ can be chosen arbitrarily small. In particular, the spectral curve is not branched over the point $\xi = \infty$.

PROOF. Let us consider the deformation of the Higgs field in terms of the coordinate $\zeta = \xi^{-1}$ in $\hat{U}_\infty$. As we see from (4.33) and (4.35), it is given by

$$\theta_\zeta = \zeta\theta - \frac{1}{2}\mathrm{d}z \wedge .$$

Notice that as $\zeta \to 0$, the first term on the right-hand side in a fixed point $z \in \mathbf{CP}^1 \smallsetminus P$ becomes insignificant, and $\theta_\zeta(z)$ converges to $-1/2\mathrm{d}z\wedge$. Therefore, for $|\zeta|$ sufficiently small, all zeros of $det(\theta_\xi)$ are in a neighborhood of P. In order to determine the asymptotic of this convergence, remember that in a holomorphic trivialisation of E in some neighborhood of p_j the Higgs field is equal to the model (1.28) up to terms in $O(z - p_j)$. As in the case $\xi \to \xi_l$, the solutions are close to those of the diagonal model $det(diag(\theta_\zeta(\tilde{q}))) = 0$ (see Claim 4.24). This equation is

$$\Pi_{k=1}^r \left(\frac{\zeta\lambda_k^j}{\tilde{q} - p_j} - \frac{1}{2}\right) = 0.$$

The solutions $\tilde{q}_k^j(\zeta)$ are clearly given by

$$\tilde{q}_k^j(\zeta) = p_j + 2\zeta\lambda_k^j = p_j + 2\frac{\lambda_k^j}{\xi}.$$

Here the upper index of the solution stands for the point $p_j \in P$ it converges to, and the lower index $k \in \{r_j + 1, \ldots, r\}$ is determined by the extension of the cokernel of θ_ζ at the point. An application of Rouché's theorem gives again the claim.

Finally, Σ is not ramified at $\xi = \infty$ because this would imply that at least two of the $q_k(\xi)$ admit the same Puiseux expansion, which is impossible because of (4.37) and (1) of Hypothesis 1.28.   □

Furthermore, by Claim 4.16 the points of Σ_ξ define a multi-valued meromorphic function in the variable ξ near infinity. Let $q_k^j(\xi) \in \Sigma_\xi$ be such a holomorphically varying zero of $det(\theta_\xi)$, and suppose it converges

to $p_j \in P$ as $\xi \to \infty$. We can let the index k to vary from $r_j + 1$ to r. Consider the diagram

$$
\begin{array}{ccc}
& \Sigma & \\
& \Big\downarrow {\scriptstyle \jmath} & \\
& \mathbf{CP}^1 \times \widehat{\mathbf{CP}}^1 & \\
{\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\
\mathbf{CP}^1 & & \widehat{\mathbf{CP}}^1
\end{array}
$$

where $\jmath$ is inclusion and the two other arrows are canonical projections. In order to define a local holomorphic section of the transformed bundle, we need to choose elements of $coKer(\theta_\xi(q_k^j(\xi)))$ for all ξ, such that they depend holomorphically with ξ. It is clear that this is equivalent to choose a local holomorphic section ψ of $\jmath^* \pi_1^* \mathcal{F}$ over the branch $(q_k^j(\xi), \xi)$ near the point (p_j, ∞) such that for all ξ, we have $\psi(q_k^j(\xi), \xi) \in coKer(\theta_\xi(q_k^j))$. Since any local section of $\mathcal{F}$ near p_j multiplied by $(z - p_j)$ is a local section of the sheaf $\mathcal{E} \otimes \mathrm{d}z$, the section $(q_k^j(\xi) - p_j)\psi$ of $\jmath^* \pi_1^* \mathcal{F}$ near (p_j, ∞) is in fact a local holomorphic section of $\jmath^* \pi_1^* (\mathcal{E} \otimes \mathrm{d}z)$ on the branch $(q_k^j(\xi), \xi)$ of the spectral curve $\Sigma \subset \mathbf{CP}^1 \times \widehat{\mathbf{CP}}^1$. Furthermore, because of Claim 4.27, $(q_k^j(\xi), \xi) \mapsto q_k^j(\xi)$ is a simple cover near p_j without branching. In particular, for all q sufficiently close to p_j there exists a unique $\xi(q)$ such that $q = q_k^j(\xi(q))$. Therefore, $(q_k^j(\xi) - p_j)\psi(q_k^j(\xi), \xi)$ is the lift from $\mathbf{CP}^1$ of a section $\varsigma_k^j(z)\mathrm{d}z$ of $\mathcal{E} \otimes \mathrm{d}z$ in a neighborhood of p_j, such that for all q we have

$$(4.38) \qquad \varsigma_k^j(q)\mathrm{d}z \in coKer(\theta_{\xi(q)}(q)).$$

In particular, $\varsigma_k^j(p_j)\mathrm{d}z \in coKer(\theta_\infty(p_j)) = E_{sing} \otimes \mathrm{d}z$, as it can easily be checked using formula (4.35). Conversely, we may consider any section $\varsigma_k^j(z)$ satisfying (4.38), lift $\varsigma_k^j(z)\mathrm{d}z$ to a section of $\jmath^* \pi_1^* (\mathcal{E} \otimes \mathrm{d}z)$, and divide the result by $q - p_j$ to obtain ψ. Fix now for all $k = \{r_j + 1, \dots, r\}$ a section ς_k^j satisfying (4.38). All that we have said above motivates the definition:

$$(4.39) \qquad v_k^j(z, \xi) = \chi_{\varepsilon_0 \xi^{-1}}(z - q_k^j(\xi)) \frac{\varsigma_k^j(z)}{z - p_j} \otimes s_0(\xi),$$

where we recall again from the proof of Proposition 4.22 that $\chi_{\varepsilon_0 \xi^{-1}}$ is a bump-function over the disk of radius $\varepsilon_0/|\xi|$. Remark that evaluation of $v_k^j(z, \xi)\mathrm{d}z$ in $z = q_k(\xi)$ is by definition in the cokernel of θ_ξ. Also,

as in the case of logarithmic singularities, for all ξ close to infinity, let $t_k^j(z, \xi)$ be the unique section of E satisfying the equation (4.31) for all z, in other words such that $D_\xi''(v_k^j(z, \xi)\mathrm{d}z, t_k^j(z, \xi)\mathrm{d}\bar{z}) = 0$. A holomorphic trivialisation of ${}^i\hat{\mathcal{E}}$ at infinity is then given by the D_ξ''-harmonic representatives $\hat{\sigma}_k^\infty(\xi)$ of the couples $(v_k^j(z, \xi)\mathrm{d}z, t_k^j(z, \xi)\mathrm{d}\bar{z})$ for all $k = \{r_j + 1, \ldots, r\}$ and all $j = \{1, \ldots, n\}$.

4.5. Singularities of the transformed Higgs field

In this part, we describe the eigenvalues of the singular parts of the transformed Higgs field $\hat{\theta}^H$ at the singularities. This establishes points (4), (6) and (7) of Theorem 1.32.

4.5.1. The case of a logarithmic singularity.

Recall from (4.13) that the transformed Higgs field is defined as multiplication by the coordinate $-z/2$ of a harmonic spinor, followed by projection onto harmonic forms.

LEMMA 4.28. *The set of eigenvalues of the transformed Higgs field* $\hat{\theta}^H$ *on the fiber* $\hat{E}_\xi^H$ *(with multiplicities) is equal to* $-\Sigma_\xi/2$ *(with multiplicities), where* Σ_ξ *is the set of zeros of* $\det(\theta_\xi)$.

PROOF. Let a cohomology class in the space $\hat{E}_\xi^H = H^1(\mathcal{D}_\xi)$ (see 4.23) be represented by 1-forms $(v(\xi)\mathrm{d}z, t(\xi)\mathrm{d}\bar{z}) \in (\Omega^{1,0} \oplus \Omega^{1,0}) \otimes E$. Since this spinor is not harmonic, first of all we need a technical result:

CLAIM 4.29. *Let* $(v(\xi)dz, t(\xi)d\bar{z}) \in (\Omega^{1,0} \oplus \Omega^{1,0}) \otimes E$ *be annihilated by* D_ξ''. *Then we have*

$$\hat{\pi}_\xi^H(z\hat{\pi}_\xi^H(v(\xi)dz, t(\xi)d\bar{z})) = \hat{\pi}_\xi^H(z(v(\xi)dz, t(\xi)d\bar{z})).$$

In words, the action of the Higgs field can be computed on any representative section in $Ker(D_\xi'')$.

PROOF. This is straightforward: we need to show

$$\hat{\pi}_\xi^H(z(\mathrm{Id} - \hat{\pi}_\xi^H)(v(\xi)\mathrm{d}z, t(\xi)\mathrm{d}\bar{z})) = 0,$$

which is equivalent to

$$z\slashed{\partial}_\xi'' G_\xi(\slashed{\partial}_\xi'')^*(v(\xi)\mathrm{d}z, t(\xi)\mathrm{d}\bar{z}) \perp \hat{E}_\xi^H.$$

Now the only thing to remark is that if $(v(\xi)\mathrm{d}z, t(\xi)\mathrm{d}\bar{z}) \in Ker(D_\xi'')$, then this implies that

$$(\slashed{\partial}_\xi'')^*(v(\xi)\mathrm{d}z, t(\xi)\mathrm{d}\bar{z}) = (D_\xi'')^*(v(\xi)\mathrm{d}z, t(\xi)\mathrm{d}\bar{z}) \in \Omega^0 \otimes E,$$

and by diagonality of G_ξ with respect to the decomposition $S^+ \otimes E = (\Omega^0 \otimes E) \oplus (\Omega^2 \otimes E)$ (see Lemma 2.24), also

$$G_\xi(\bar{\phi}''_\xi)^*(v(\xi)\mathrm{d}z, t(\xi)\mathrm{d}\bar{z}) \in \Omega^0 \otimes E.$$

Therefore we have

$$\bar{\phi}''_\xi G_\xi(\bar{\phi}''_\xi)^*(v(\xi)\mathrm{d}z, t(\xi)\mathrm{d}\bar{z}) = D''_\xi G_\xi(D''_\xi)^*(v(\xi)\mathrm{d}z, t(\xi)\mathrm{d}\bar{z}),$$

and we conclude using the commutation relation

$$[z, D''_\xi] = 0$$

combined with $Im(D''_\xi) \perp \hat{E}^H_\xi$.   □

The proof of the lemma is now immediate: via the map (4.28),

$$\Psi^{-1}_\xi(z(v(\xi)\mathrm{d}z, t(\xi)\mathrm{d}\bar{z})) = (q \cdot eval_q v(\xi))_{q \in \Sigma_\xi}$$

multiplication by z goes over to multiplication by q in the point $q \in \Sigma_\xi$, and via (4.27) this is then re-transformed into multiplication by the constant q on the component of $v(\xi)$ localized near q.   □

THEOREM 4.30. *The eigenvalues of the transformed Higgs field $\hat{\theta}^H$ have first-order poles in the points of $\hat{P}$. Furthermore, the non-vanishing eigenvalues of its residue in the puncture ξ_l are equal to $\{-\lambda^\infty_{1+a_l}, \ldots, -\lambda^\infty_{a_{l+1}}\}$, where $\{\lambda^\infty_{1+a_l}, \ldots, \lambda^\infty_{a_{l+1}}\}$ are the eigenvalues of the residue of the original Higgs field θ at infinity, restricted to the eigenspace of A corresponding to the eigenvalue ξ_l.*

PROOF. As we have seen in (4.29), the point $q_k(\xi) \in \Sigma_\xi$ converges to infinity at the first order with $2\lambda^\infty_k(\xi - \xi_l)^{-1}$ as $\xi \to \xi_l$, where $k \in \{1+a_l, \ldots, a_{l+1}\}$ is an index such that the eigenvalue λ^∞_k of the residue term of θ at infinity appears in the eigenspace of the second order term A corresponding to the eigenvalue ξ_l. By Lemma 4.28, the transformed Higgs field has a logarithmic singularity at ξ_l, and the corresponding residue is $-\lambda^\infty_k$.   □

4.5.2. The case of infinity. We wish to show the following.

THEOREM 4.31. *The transformed Higgs field has a second order singularity at infinity. The set of eigenvalues of its leading order term is $\{-p_1/2, \ldots, -p_n/2\}$, where $\{p_1, \ldots, p_n\} = P$ is the set of punctures of the original Higgs bundle. The multiplicity of the eigenvalue $-p_j/2$ is equal to $r - r_j = rk(Res(\theta, p_j))$. The set of eigenvalues of the residue*

of the transformed Higgs field restricted to the eigenspace of the second-order term corresponding to the eigenvalue $-p_j/2$ is $\{-\lambda_k^j\}_{k\in\{r_j+1,\dots,r\}}$.

PROOF. In Claim 4.27 we have proved that as $\zeta \to 0$, all zeros of $det(\theta_\zeta)$ must converge to one of the points of P. Furthermore, the expansion of a spectral point q_k converging to p_j is (4.37). By Lemma 4.28, on the corresponding components $\hat{\theta}^H$ is just multiplication by $-\Sigma_\xi d\xi/2$. Hence, we see that the eigenvalues of the leading-order term of the transformed Higgs field are equal to $\{-p_j/2\}_{j=1,\dots,n}$, while those of its first-order term are $\{-\lambda_k^j\}_{j=1,\dots,n;k=r_j+1,\dots,r}$.   □

4.6. Parabolic weights

Here we compute the parabolic weights of the transformed Higgs bundle with respect to the induced extension.

4.6.1. The case of infinity.

THEOREM 4.32. *The parabolic weight of the extension ${}^i\hat{\mathcal{E}}$ of the transformed Higgs bundle at infinity described in Subsection 4.4.2, restricted to the eigenspace of $\hat{\theta}$ corresponding to the eigenvalue $-p_j/2$ of its second order term and the eigenvalue $-\lambda_k^j$ of its residue is equal to $-1 + \alpha_k^j$, where α_k^j is the parabolic weight on the λ_k^j-eigenspace of the residue of the original Higgs bundle at p_j.*

PROOF. We prove the statement in two steps. In the first one, we show that it is true supposing the original Higgs bundle only has one logarithmic point of a precise form. In the second one, we show how the case with only one logarithmic point and the exponential decay results of Section 2.5 imply the general case.

Step 1. Let us first suppose that the set of logarithmic singularities is reduced to a single point p_1, that we may take to be 0 without restricting generality. Furthermore, we suppose that E is a holomorphically trivial bundle over $\mathbf{C}$ and that in a global holomorphic trivialisation $\{\sigma_k\}$ the Higgs field is equal to

$$\theta = diag\left(\frac{\lambda_k}{z}\right)_{k=1,\dots,r} dz$$

and the metric is just

(4.40) $$h(\sigma_k, \sigma_k) = |z|^{2\alpha_k}.$$

This defines a parabolic Higgs bundle with weights α_k at 0 and $-\alpha_k$ at infinity, the field having deformation

$$(4.41) \qquad \theta_\xi = diag\left(\frac{\lambda_k}{z} - \frac{\xi}{2}\right)_{k=1,\ldots,r} dz$$

and the D''-operator

$$(4.42) \qquad D''_\xi = \bar\partial + diag\left(\lambda_k\frac{dz}{z} - \frac{\xi}{2}dz\right)_{k=1,\ldots,r}.$$

Recall from Subsection 4.4.2 that a representative $(v_\xi dz, t_\xi d\bar z)$ of any spinor ψ_ξ is supported in the finite collection of disks $\cup_{q(\xi)\in\Sigma_\xi}\Delta(q(\xi), \varepsilon_0|\xi|^{-1})$. By Claim 4.27, the points $q(\xi)$ are given by

$$(4.43) \qquad q_k(\xi) = \frac{2\lambda_k}{\xi}.$$

Define a family of homotheties indexed by $\xi \in \hat{\mathbf{C}} \smallsetminus \hat{P}$

$$h_\xi : \mathbf{C} \longrightarrow \mathbf{C}$$

$$(4.44) \qquad w \mapsto z = \frac{w}{\xi};$$

in such a way that

$$h_\xi^{-1}(0) = 0$$

$$(4.45) \qquad h_\xi^{-1}(q_k(\xi)) = 2\lambda_k \qquad \text{for} \quad k = r_1, \ldots, r.$$

Therefore, this corresponds to a family of coordinate changes $z \leftrightarrow w$ in the plane, such that the position of the zeros of the Higgs field θ_ξ after applying h_ξ^{-1} is constant (the $2\lambda_k$ for $k = r_1, \ldots, r$), as well as that of the poles (0 and ∞). Moreover, $dz = \xi^{-1}dw$ implies

$$(4.46) \qquad h_\xi^*\theta_\xi = diag\left[\lambda_k\frac{dw}{w} - \frac{1}{2}dw\right]_{k=1,\ldots,r},$$

and so

$$(4.47) \qquad h_\xi^*D''_\xi = \bar\partial + diag\left[\lambda_k\frac{dw}{w} - \frac{1}{2}dw\right]_{k=1,\ldots,r},$$

where $\bar\partial$ stands this time for the Dolbeault operator with respect to the w-coordinate. The crucial observation is that this operator is independent of ξ. On the other hand, remark that the Euclidean metric on the

base space and the fiber metric (4.40) behave under these coordinate changes as

$$(4.48) \qquad\qquad (h_\xi)_* |\mathrm{d}w|^2 = |\xi|^2 |\mathrm{d}z|^2$$

$$(4.49) \qquad\qquad |\sigma_k(z)|^2 = |\xi|^{-2\alpha_k} |w|^{2\alpha_k}.$$

In other words, if we denote by $h^{(w)}$ the model hermitian metric on $h_\xi^* E$ equal in the basis $h_\xi^* \sigma_k$ to

$$h^{(w)} = diag(|w|^{2\alpha_k}),$$

then the homotheties h_ξ induce a family of tautological isomorphisms of Hermitian fiber bundles

$$(4.50) \qquad\qquad (h_\xi^* E, h^{(w)}) \longrightarrow (E, h)$$

$$(h_\xi^* \sigma_k)(w) \mapsto |\xi|^{\alpha_k} \sigma_k(z).$$

We deduce from (4.48) that in the basis $h_\xi^* \sigma_k$ the pull-back $h_\xi^* \Delta_\xi$ of the Laplacian of the Dirac operator $\slashed{\partial}_\xi''$ has the form

$$(4.51) \qquad\qquad |\xi|^2 \left[\Delta + diag \left| \frac{\lambda_k}{w} - \frac{1}{2} \right|^2_{k=1,\dots,r} \right],$$

where Δ stands for the usual Laplace operator on functions with respect to the metric $|\mathrm{d}w|^2$. The operator $\Delta^{(w)}$ between brackets in this formula is a bounded operator from the weighted Sobolev space $H^2(S^+ \otimes E, |\mathrm{d}w|^2)$ to $L^2(S^+ \otimes E, |\mathrm{d}w|^2)$. The weight at 0 is determined by the condition that for a section $u \in H^2$ we have $u/|w|^2 \in L^2$, and this gives therefore exactly the maximal domain of $\Delta^{(w)}$ (see Theorem 2.22). We infer that the pull-back $h_\xi^* G_\xi$ of the Green's operator of Δ_ξ is

$$(4.52) \qquad\qquad |\xi|^{-2} G^{(w)},$$

where $G^{(w)}$ is the inverse of $\Delta^{(w)}$. It also follows from Theorem 2.22 that $G^{(w)}$ is a bounded linear operator from $L^2(S^+ \otimes E, |\mathrm{d}w|^2)$ to $H^2(S^+ \otimes E, |\mathrm{d}w|^2)$. Because $\Delta^{(w)}$ is diagonal in the basis σ_k, the same is true for $G^{(w)}$. Remark that the pull-backs $h_\xi^* \hat{\pi}_\xi$ of the orthogonal projections onto Δ_ξ-harmonic spinors are all equal to the orthogonal projection $\hat{\pi}^{(w)}$ onto $\Delta^{(w)}$-harmonic spinors: indeed, the conformal factor $|\xi|^2$ in (4.51) changes neither the space of harmonic spinors nor the orthogonal projection operator onto them. In particular, since $\Delta^{(w)}, G^{(w)}$ and h are diagonal in the basis σ_k, the same thing is true for all $\hat{\pi}_\xi$.

Now notice that by the definition of the $\bar{\partial}^{\hat{\varepsilon}}$-holomorphic extension to infinity of the transformed bundle given in (4.39) and via the identification (4.50), the sections $|\xi|^{\alpha_k} h_\xi^*(v_k(z,\xi)\mathrm{d}z)$ (modulo the value of the section s_0 of $\mathcal{O}_{\widehat{\mathbf{CP}}^1}(1)$) coincide: indeed,

$$|\xi|^{\alpha_k} \chi_{\varepsilon_0/\xi}(z - q_k(\xi))\sigma_k(z)\frac{\mathrm{d}z}{z} = \chi_{\varepsilon_0}\left(w - 2\lambda_k\right)(h_\xi^*\sigma_k)(w)\frac{\mathrm{d}w}{w}.$$

It then follows from formula (4.47) together with the definition (4.31) that the coefficient of s_0 in $|\xi|^{\alpha_k} h_\xi^* t_k(z,\xi)\mathrm{d}\bar{z}$ is also independent of ξ. From the fact that the projections $\hat{\pi}_\xi$ are also constant, we deduce that the coefficient of s_0 in the pull-back

(4.53) $$(h_\xi^* \hat{\sigma}_k)(w,\xi) = |\xi|^{\alpha_k} \hat{\sigma}_k^\infty(z,\xi)$$

of the spinors $|\xi|^{\alpha_k} \hat{\sigma}_k^\infty(z,\xi)$ representing $|\xi|^{\alpha_k}(v_k(z,\xi)\mathrm{d}z, t_k(z,\xi)\mathrm{d}\bar{z})$ does not depend on ξ. Therefore, denoting by $f_k(z,\xi)$ the coefficient of s_0 in $\hat{\sigma}_k^\infty(z,\xi)$ and by $(h_\xi^* f_k)(w,\xi)$ the coefficient of s_0 in $(h_\xi^* \hat{\sigma}_k^\infty)(w,\xi)$, we see by invariance of the L^2-norm of 1-forms by conformal coordinate change that

$$\int_{\mathbf{C}} |f_k(z,\xi)|^2_{h,|dz|^2}|\mathrm{d}z|^2 = |\xi|^{-2\alpha_k} \int_{\mathbf{C}} |(h_\xi^* f_k)(w,\xi)|^2_{h^{(w)},|dw|^2}|\mathrm{d}w|^2,$$

for all ξ, with the integral on the right-hand side a constant independent of ξ. On the other hand, recall from (4.32) that on the affine chart $\hat{U}_0$ of $\mathbf{CP}^1$ we have $s_0(\xi) = \xi$. Observe also that the transformed Hermitian metric $\hat{h}$ is defined in the chart $\hat{U}_0$, and that for any harmonic spinor f we have

$$\hat{h}(\xi f, \xi f) = |\xi|^2 \hat{h}(f,f) = |\zeta|^{-2}\hat{h}(f,f)$$

with $\zeta = \xi^{-1}$ the local coordinate centered at 0 of the singularity at infinity. This means that the effect on the parabolic weights of multiplying by s_0 is adding -1. On the other hand, the $-\lambda_k$-eigenspace of the residue of the transformed Higgs bundle at infinity is spanned by $\hat{\sigma}_k^\infty$. From all that has been said above, we deduce

(4.54) $$\hat{h}(\hat{\sigma}_k^\infty, \hat{\sigma}_k^\infty) = M|\zeta|^{-2+2\alpha_k},$$

where M is independent of ξ; in different terms, that the parabolic weight of the transformed Higgs bundle at infinity on the $-\lambda_k$-eigenspace of the residue is equal to $-1 + \alpha_k$.

Step 2. Starting from now, we drop the assumption that the set of logarithmic singularities is reduced to a point. In this part, we patch together solutions to local problems provided by Step 1, and use the results of Section 2.5 to estimate the defect of these patched sections to be solutions of the global problem. We find that the interaction between solutions to local problems near different punctures is small as $|\xi|$ gets large.

Let $(\bar{\partial}^{\mathcal{E}}, \theta)$ be a Higgs bundle with some logarithmic singularities $P = \{p_1, \ldots, p_n\}$. In a holomorphic trivialisation $\{\sigma_k^j\}_{k=1}^r$ near each one of these points, up to terms in $O(1)\mathrm{d}z$, the Higgs field has the form

$$(4.55) \qquad \theta^j = \frac{A^j}{z - p_j}\mathrm{d}z,$$

where the A^j are some diagonal matrices as in (1.1). The deformation of these local models is

$$\theta_\xi^j = \left[\frac{A^j}{z - p_j} - \frac{\xi}{2} \right] \mathrm{d}z,$$

and similarly the deformation of the local D''-operators $(D'')^j$ is

$$(D_\xi'')^j = \bar{\partial}^{\mathcal{E}} + \theta_\xi$$
$$= \bar{\partial}^{\mathcal{E}} + \left[\frac{A^j}{z - p_j} - \frac{\xi}{2} \right] \mathrm{d}z,$$

Finally, that of the Dirac operator $\slashed{\partial}^j = (D'')^j - ((D'')^j)^*$ is

$$\slashed{\partial}_\xi^j = (D_\xi'')^j - ((D_\xi'')^j)^*,$$

adjoint being taken relative to the harmonic metric corresponding to $(D'')^j$. Now for all j we can consider the extension of θ^j to a trivial bundle E^j over the whole plane by keeping the same formula (4.55) for it, endowed with the model metric

$$h^j = diag(|z - p_j|^{2\alpha_k^j})_{k=1}^r.$$

It is clear that this extension only has one regular singularity (in p_j) and an irregular one at infinity, so all the results of Step 1 hold for them. In particular, for representatives

$$(v_k^j(z, \xi)\mathrm{d}z, t_k^j(z, \xi)\mathrm{d}\bar{z})$$

as described in Subsection 4.4.2 we have a harmonic representative

$$\hat{\sigma}_k^\infty(z, \xi) \in Ker(\slashed{\partial}_\xi^j)^* \subset H^1(\mathbf{C}, S^- \otimes E^j)$$

with

$$\int_{\mathbf{C}} |\hat{\sigma}_k^\infty(z, \xi)|^2_{h^j, |dz|^2} \, |dz|^2 = |\xi|^{2-2\alpha_k^j}.$$

This growth is measured with respect to the diagonal model metric h^j; however, since the spinor $\hat{\sigma}_k^\infty$ is exponentially concentrated near p_j and here h^j is mutually bounded with the harmonic metric h of $(\mathcal{E}, \theta)$, this implies

$$(4.56) \qquad c|\xi|^{2-2\alpha_k^j} \leq \int_{\mathbf{C}} |\hat{\sigma}_k^\infty(z, \xi)|^2_{h, |dz|^2} \, |dz|^2 \leq C|\xi|^{2-2\alpha_k^j}$$

for some $0 < c < C$. Let χ^j be a cut-off function supported in a disk $\Delta(p_j, 3\varepsilon_0)$, equal to 1 on $\Delta(p_j, 2\varepsilon_0)$, such that $|\nabla \chi^j| \leq K$. Then for $\varepsilon_0 > 0$ fixed sufficiently small, the global section of $S^- \otimes E$ defined by

$$\hat{\sigma}(z, \xi) = \chi^j(z)\hat{\sigma}_k^\infty(z, \xi)$$

has a meaning, for the holomorphic trivialisation $\{\sigma_k^j\}$ is defined in $\Delta(p_j, 3\varepsilon_0)$ provided ε_0 is sufficiently small. Now notice that if $q(\xi) \to p_j$ as $\xi \to \infty$ and more precisely

$$q(\xi) = p_j + \frac{2\lambda_k^j}{\xi} + O(|\xi|^{-2}),$$

in other words on the component of the transformed bundle with eigenvalue of the second-order part of $\hat{\theta}$ at infinity equal to $-p_j/2$ and eigenvalue of the residue of $\hat{\theta}$ at infinity equal to $-\lambda_k^j$, the holomorphic extension ς_k^j of the cokernel has as parabolic weight the α_k^j corresponding to the eigenspace of the eigenvalue λ_k^j of the residue of θ. Recall that the harmonic metric on the transformed side is just L^2-metric of the Δ_ξ-harmonic representative with respect to the harmonic metric h of the original Higgs bundle. The statement of the theorem will therefore follow once we prove that the harmonic representative of $\hat{\sigma}(z, \xi)$ satisfies the inequality

$$(4.57) \qquad c|\xi|^{2-2\alpha_k^j} \leq \int_{\mathbf{C}} |\hat{\pi}_\xi \hat{\sigma}(z, \xi)|^2_{h, |dz|^2} \, |dz|^2 \leq C|\xi|^{2-2\alpha_k^j}.$$

for some $0 < c < C$. Our first aim is to prove the following.

LEMMA 4.33. *There exists $\delta > 0$ and $K > 0$ such that for $|\xi|$ sufficiently large the inequality*

$$\left\| \hat{\rlap{/}\partial}_\xi^* \hat{\sigma}(\xi) \right\|^2_{L^2(\mathbf{C})} \leq K|\xi|^{2-2\delta} \left\| \hat{\sigma}(\xi) \right\|^2_{L^2(\mathbf{C})}$$

holds.

PROOF. Covering the annulus centered at p_j of radii $2\varepsilon_0$ and $2R_0$ by a finite number of disks of radius ε_0, we deduce from Lemmas 2.30 and 2.31 that the $\partial\!\!\!/_\xi^j$-harmonic spinor $\hat{\sigma}_k^\infty(z, \xi)$ is concentrated in H^1-norm, up to a factor decreasing exponentially with $|\xi|$, in the disk $\Delta(p_j, 2\varepsilon_0)$. In particular, it is concentrated up to an exponentially decreasing factor in the same disk in L^2-norm as well. Denoting by $\cdot$ Clifford multiplication, we have the estimation

$$\int_{\mathbf{C}} \left| \partial\!\!\!/_\xi^*(\chi^j(z)\hat{\sigma}_k^\infty(z, \xi)) \right|^2 |\mathrm{d}z|^2 \leq \int_{\mathbf{C}} \left| \chi^j(z)\partial\!\!\!/_\xi^*\hat{\sigma}_k^\infty(z, \xi) \right|^2 |\mathrm{d}z|^2$$
$$+ \int_{\mathbf{C}} \left|(\nabla\chi^j)(z) \cdot \hat{\sigma}_k^\infty(z, \xi)\right|^2 |\mathrm{d}z|^2$$
$$\leq \int_{\Delta(p_j, 3\varepsilon_0)} \left| \partial\!\!\!/_\xi^*\hat{\sigma}_k^\infty(z, \xi) \right|^2 |\mathrm{d}z|^2$$
$$+ K \int_{\Delta(p_j, 3\varepsilon_0) \smallsetminus \Delta(p_j, 2\varepsilon_0)} \left|\hat{\sigma}_k^\infty(z, \xi)\right|^2 |\mathrm{d}z|^2.$$

Again, by Lemma 2.30 the second integral on the right-hand side is bounded by an exponentially decreasing multiple of $\|\hat{\sigma}_k^\infty(z, \xi)\|_{L^2(\mathbf{C})}^2$ as $|\xi| \to \infty$. Therefore, we only need to treat

$$\left\| \partial\!\!\!/_\xi^*\hat{\sigma}_k^\infty(z, \xi) \right\|_{L^2(\Delta(p_j, 3\varepsilon_0))}^2 .$$

Remark that by hypothesis,

$$(\partial\!\!\!/_\xi^j)^*\hat{\sigma}_k^\infty(z, \xi) = 0,$$

so we have

$$\partial\!\!\!/_\xi^*\hat{\sigma}_k^\infty(z, \xi) = \left[\partial\!\!\!/_\xi^* - (\partial\!\!\!/_\xi^j)^* \right] \hat{\sigma}_k^\infty(z, \xi).$$

This is then bounded by

$$\hat{\sigma}_k^\infty(z, \xi)O(|z - p_j|^{-1+\delta}),$$

where $O(|z - p_j|^{-1+\delta})$ stands for a term bounded from above by a constant (independent of ξ) times $|z - p_j|^{-1+\delta}$, because $\partial\!\!\!/_\xi^*$ and $(\partial\!\!\!/_\xi^j)^*$ are Dirac operators having the same local model at the puncture and their difference is clearly independent of ξ. In order to study this quantity, we make use of the coordinate $w = \xi(z - p_j)$ analogously to that introduced in (4.44). Under this coordinate change, the disk $\Delta(p_j, 3\varepsilon_0)$

goes into the (varying) disk $\Delta(0, 3\varepsilon_0|\xi|)$. Hence, we need to prove

$$\int_{\Delta(0,3\varepsilon_0|\xi|)} |w|^{-2+2\delta}|\xi|^{2-2\delta} \left| (h_\xi^*\hat{\sigma}_k^\infty)(w,\xi) \right|^2_{|dz|^2,h} |\xi|^{-2}|dw|^2$$

$$\leq K|\xi|^{2-2\delta} \int_{\mathbf{C}} \left| (h_\xi^*\hat{\sigma}_k^\infty)(w,\xi) \right|^2_{|dz|^2,h} |\xi|^{-2}|dw|^2$$

Recall from (4.53) that in the coordinate w the spinors $|\xi|^{-\alpha_k^j} h_\xi^* \hat{\sigma}_k^\infty$ are independent of ξ. Therefore this boils down to

$$\int_{\Delta(0,3\varepsilon_0|\xi|)} |w|^{-2+2\delta} \left| (h_\xi^*\hat{\sigma}_k^\infty)(w) \right|^2_{|dz|^2} |dw|^2$$

$$(4.58) \qquad\qquad \leq K \int_{\mathbf{C}} \left| (h_\xi^*\hat{\sigma}_k^\infty)(w) \right|^2_{|dz|^2} |dw|^2$$

for a suitable constant $K > 0$. Because

$$(h_\xi^*\hat{\sigma}_k^\infty)(w) \in H^1(\mathbf{C}),$$

in particular we have

$$(h_\xi^*\hat{\sigma}_k^\infty)(w) \in L^2(\mathbf{C}),$$

and also

$$\frac{1}{w}(h_\xi^*\hat{\sigma}_k^\infty)(w) \in L^2_{loc}.$$

near the origin. This implies $|w|^{-1+\delta}(h_\xi^*\hat{\sigma}_k^\infty)(w) \in L^2(\mathbf{C})$. Therefore,

$$K = 2\frac{\left\| |w|^{-1+\delta}(h_\xi^*\hat{\sigma}_k^\infty)(w) \right\|^2_{L^2(\mathbf{C})}}{\left\| (h_\xi^*\hat{\sigma}_k^\infty)(w) \right\|^2_{L^2(\mathbf{C})}}$$

has the desired property. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

The lemma has the following consequence.

LEMMA 4.34. *As $|\xi| \to \infty$, we have the estimate*

$$\left| \|\hat{\sigma}(\xi)\|^2_{L^2} - \|\hat{\pi}_\xi^H \hat{\sigma}(\xi)\|^2_{L^2} \right| \leq K|\xi|^{-2\delta} \|\hat{\sigma}(\xi)\|^2_{L^2}$$

with $K > 0$ independent of ξ.

PROOF. It is sufficient to bound

$$\left\| \hat{\sigma}(\xi) - \hat{\pi}_\xi^H \hat{\sigma}(\xi) \right\|^2_{L^2}$$

as in the lemma. The $\displaystyle{\not}\partial_\xi^*$-harmonic representative $\hat{\pi}_\xi^H \hat{\sigma}(\xi)$ of $\hat{\sigma}(\xi)$ is given by the formula

$$(\mathrm{Id} - {\not}\partial_\xi G_\xi {\not}\partial_\xi^*)\hat{\sigma}(\xi),$$

so the difference with $\hat{\sigma}(\xi)$ itself is

$$\slashed{\partial}_\xi G_\xi \slashed{\partial}_\xi^* \hat{\sigma}(\xi).$$

Since for any positive spinor φ the estimation

$$\|\slashed{\partial}_\xi \varphi\|_{L^2(\mathbf{C})}^2 \leq K \|\varphi\|_{H^1(\mathbf{C})}^2 + K|\xi|^2 \|\varphi\|_{L^2(\mathbf{C})}^2$$

holds, we deduce that

$$\left\|\slashed{\partial}_\xi G_\xi \slashed{\partial}_\xi^* \hat{\sigma}(\xi)\right\|_{L^2(\mathbf{C})}^2 \leq K \left\|G_\xi \slashed{\partial}_\xi^* \hat{\sigma}(\xi)\right\|_{H^1(\mathbf{C})}^2 + K|\xi|^2 \left\|G_\xi \slashed{\partial}_\xi^* \hat{\sigma}(\xi)\right\|_{L^2(\mathbf{C})}^2.$$

Lemma 2.25 implies that both terms on the right-hand side are bounded from above by

$$K|\xi|^{-2} \left\|\slashed{\partial}_\xi^* \hat{\sigma}(\xi)\right\|_{L^2(\mathbf{C})}^2.$$

We conclude by Lemma 4.33.   □

We can now finish the proof of Theorem 4.32: as $|\xi|$ goes to infinity, by Lemma 4.34, we have

$$\frac{\left\|\hat{\pi}_\xi^H \hat{\sigma}(\xi)\right\|_{L^2}^2}{\|\hat{\sigma}(\xi)\|_{L^2}^2} \longrightarrow 1.$$

In words, the norm of the harmonic representative of the spinor $\hat{\sigma}(z,\xi)$ is asymptotically equal to the norm of $\hat{\sigma}(z,\xi)$ itself. On the other hand, as it has already been remarked in the proof of Lemma 4.33, we have

$$\frac{\|\hat{\sigma}(z,\xi)\|_{L^2(\mathbf{C},h)}^2}{\|\hat{\sigma}_k^\infty(z,\xi)\|_{L^2(\mathbf{C},h)}^2} \longrightarrow 1$$

exponentially as $\xi \to \infty$. Finally, by (4.56) the L^2-norm of the spinors $\hat{\sigma}_k^\infty(z,\xi)$ as measured by the harmonic metric h satisfy

$$(4.59) \qquad c|\xi|^{2-2\alpha_k^j} \leq \|\hat{\sigma}_k^\infty(z,\xi)\|_{L^2}^2 \leq C|\xi|^{2-2\alpha_k^j}$$

for some $0 < c < C$, where α_k^j is a parabolic weight of the original Higgs bundle at the point p_j. All this then implies (4.57), so it follows that the parabolic weight of the transformed Higgs bundle on the given component is equal to $\alpha_k^j - 1$, as it was stated in the theorem.   □

4.6.2. The case of logarithmic singularities. Next we compute the parabolic weights at a puncture ξ_l corresponding to the extension of the holomorphic structure of $\hat{\mathcal{E}}$ given in Subsection 4.4.1.

Explicitly, here is the result we wish to show.

THEOREM 4.35. *The parabolic weight of the extension $^i\hat{\mathcal{E}}$ of the transformed Higgs bundle at the puncture ξ_l, restricted to the $-\lambda_k^\infty$-eigenspace of the residue of the transformed Higgs field (here $k \in \{1 + a_l, \ldots, a_{l+1}\}$) is equal to $-1 + \alpha_k^\infty$, where α_k^∞ is the parabolic weight of the original Higgs field at infinity, restricted to the ξ_l-eigenspace of the second-order term and the λ_k^∞-eigenspace of the first-order term of the polar part of the Higgs field.*

PROOF. We follow the proof of Theorem 4.32. Again, we divide the proof into two steps according to the number of distinct eigenvalues ξ_l of the second order term of D at infinity. Recall that some of the spectral points $q_k \in \Sigma_\xi$ converge to infinity as $\xi \to \xi_l$, whereas others remain bounded.

Step 1. First we suppose that $n' = 1$, that is to say A is a simple diagonal matrix, and that in a global holomorphic basis $\{\sigma_k^\infty\}$ the Higgs field has is of the form

$$\theta = \frac{\xi_1}{2}\mathrm{d}z + diag(\lambda_k^\infty)\frac{\mathrm{d}z}{z}$$

with one regular singularity in 0 and an irregular one at infinity, and finally the harmonic metric is

(4.60) $$h^\infty = diag(|z|^{-2\alpha_k^\infty})_{k=1}^r.$$

This induces a parabolic structure on $\mathcal{E}$ with weights $-2\alpha_k^\infty$ at 0 and $2\alpha_k^\infty$ at infinity. The deformed field is

$$\theta_\xi = \frac{\xi_1 - \xi}{2}\mathrm{d}z + diag(\lambda_k^\infty)\frac{\mathrm{d}z}{z},$$

and the spectral points are

$$\frac{2\lambda_k}{\xi - \xi_1}.$$

Making the coordinate change

$$h_\xi : \mathbf{C} \longrightarrow \mathbf{C}$$

(4.61) $$w \mapsto z = \frac{w}{\xi - \xi_1}$$

the field writes

(4.62) $$\theta_\xi = -\frac{1}{2}\mathrm{d}w + diag(\lambda_k^\infty)\frac{\mathrm{d}w}{w}.$$

The Euclidean metric $|dz|^2$ on the base and the fiber metric h^∞ are transformed into

(4.63) $$|\xi - \xi_1|^{-2}|dw|^2$$

(4.64) $$diag(|\xi - \xi_1|^{2\alpha_k^\infty}|w|^{-2\alpha_k^\infty})_{k=1}^r$$

and the position of the spectral points become simply

$$2\lambda_k,$$

independent of ξ. As in the case of the singularity at infinity, writing $h^{(w)}$ for the diagonal model metric

$$diag(|w|^{-2\alpha_k^\infty})_{k=1}^r$$

the coordinate changes induce tautological isomorphisms of Hermitian fiber bundles

(4.65) $$(h_\xi^* E, h^{(w)}) \longrightarrow (E, h^\infty)$$
$$(h_\xi^* \sigma_k)(w) \mapsto |\xi - \xi_1|^{-\alpha_k}\sigma_k(z).$$

Via this isomorphism the representatives $v_k(z, \xi)$ given in (4.30) behave as follows:

$$|\xi - \xi_1|^{-\alpha_k}v_k(z, \xi) = v_k(w),$$

which is independent of ξ, or equivalently

$$|\xi - \xi_1|^{-\alpha_k}v_k(z, \xi)(\xi - \xi_1)dz = v_k(w)dw,$$

independent of ξ. By the equation (4.31), this implies

$$|\xi - \xi_1|^{-\alpha_k}t_k(z, \xi)(\bar{\xi} - \bar{\xi}_1)d\bar{z} = t_k(w)d\bar{w},$$

independently of ξ. Exactly as in the case of the singularity at infinity, the Laplacian and the Green's operator of $\bar{\partial}_\xi^*$ in the coordinate w only depend on ξ through a conformal factor $|\xi - \xi_1|^{-2}$ and $|\xi - \xi_1|^2$ respectively, so the pull-back $h_\xi^* \hat{\pi}_\xi$ of the projection onto $\bar{\partial}_\xi^*$-harmonic spinors is independent of ξ. We deduce using invariance of the L^2-norm of 1-forms by conformal coordinate change that for the $\bar{\partial}_\xi^*$-harmonic spinor $\hat{\sigma}_k(z, \xi)$ representing the cohomology class of $(v_k(z, \xi)dz, t_k(z, \xi)d\bar{z})$ we have

$$\int_{\mathbf{C}} |\hat{\sigma}_k(z, \xi)|_{h^\infty, |dz|^2}^2 |dz|^2 = |\xi - \xi_1|^{2\alpha_k - 2}\int_{\mathbf{C}} |\hat{\sigma}_k(w)|_{h^{(w)}, |dw|^2}^2 |dw|^2,$$

where $\hat{\sigma}_k(w)$ is the harmonic spinor representing $(v_k(w)dw, t_k(w)d\bar{w})$. We see also that the integral on the right-hand side is independent of

ξ, hence we have the desired behavior giving parabolic weight $-1 + \alpha_k$ on this component.

Step 2. We drop the assumption that the second-order term A of the original Higgs field is a simple matrix. Let χ be a fixed cut-off function supported on the complementary $\mathbf{C} \setminus \Delta(0, 1/\varepsilon_0)$ of a large disk, equal to 1 on $\mathbf{C} \setminus \Delta(0, 2/\varepsilon_0)$. In $\mathbf{C} \setminus \Delta(0, 1/\varepsilon_0)$, the Higgs field is up to a perturbation

$$\theta^\infty = \frac{1}{2} A \mathrm{d}z + C \frac{\mathrm{d}z}{z}$$

with A and C diagonal matrices as in (1.31), therefore decomposes into a direct sum of problems studied in Step 1. In particular, for each such model problem with eigenvalue of the second-order term ξ_l we have harmonic spinors $\hat{\sigma}_k^l(z, \xi)$ where $k \in \{1 + a_l, \dots, a_{l+1}\}$, such that

$$\int_{\mathbf{C}} \left| \hat{\sigma}_k^l(z, \xi) \right|_{|dz|^2, h^\infty} |\mathrm{d}z|^2 = |\xi - \xi_l|^{-2 + 2\alpha_k^\infty}.$$

Again, since the harmonic metric h of the Higgs bundle $(\mathcal{E}, \theta)$ is mutually bounded with h^∞ in a neighborhood of infinity and $\hat{\sigma}_k^l$ is supported there, this implies

$$(4.66) \quad c|\xi - \xi_l|^{-2 + 2\alpha_k^\infty} \leq \int_{\mathbf{C}} \left| \hat{\sigma}_k^l(z, \xi) \right|_{|dz|^2, h} |\mathrm{d}z|^2 \leq C|\xi - \xi_l|^{-2 + 2\alpha_k^\infty}$$

for some $0 < c < C$. The section

$$\hat{\sigma}(z, \xi) = \chi(z) \hat{\sigma}_k^l(z, \xi)$$

is well-defined because the local holomorphic trivialisation σ_k^∞ of $\mathcal{E}$ is defined in $\mathbf{C} \setminus \Delta(0, 1/\varepsilon_0)$ for $\varepsilon_0 > 0$ sufficiently small. The statement of the theorem will again follow if we prove

$$(4.67) \quad c|\xi - \xi_l|^{-2 + 2\alpha_k^\infty} \leq \int_{\mathbf{C}} \left| \hat{\pi}_{\xi^H} \hat{\sigma}(z, \xi) \right|_{|dz|^2, h} |\mathrm{d}z|^2 \leq C|\xi - \xi_l|^{-2 + 2\alpha_k^\infty}$$

where $\hat{\pi}_\xi^H \hat{\sigma}(z, \xi)$ is the harmonic representative of $\hat{\sigma}(z, \xi)$. As a first step in this direction, we prove:

LEMMA 4.36. *There exists* $\delta > 0$ *and* $K > 0$ *such that for* $|\xi|$ *sufficiently large the inequality*

$$\left\| \bar{\phi}_\xi^* \hat{\sigma}(z, \xi) \right\|_{L^2(\mathbf{C})}^2 \leq K|\xi - \xi_l|^{2 + 2\delta} \left\| \hat{\sigma}(z, \xi) \right\|_{L^2(\mathbf{C})}^2$$

holds.

PROOF. We follow the proof of Lemma 4.33. We set $(D_\xi'')^\infty = \bar{\partial}^{\mathcal{E}} + \theta^\infty$ and let $\dslash_\xi^\infty$ (respectively $(\dslash_\xi^\infty)^*$) stand for its Dirac operator (respectively its adjoint). By Lemma 2.30, $\hat{\sigma}_k^l$ is supported in L^2-norm up to an exponentially decreasing factor in ξ in $\mathbf{C} \setminus \Delta(0, 1/\varepsilon_0)$. Therefore, the lemma reduces to the same estimation for $\hat{\sigma}_k^l$. Moreover, by assumption we have

$$(\dslash_\xi^\infty)^* \hat{\sigma}_k^l(z, \xi) = 0,$$

so

$$\dslash_\xi^* \hat{\sigma}_k^l(z, \xi) = [\dslash_\xi^* - (\dslash_\xi^\infty)^*]\hat{\sigma}_k^l(z, \xi).$$

The difference on the right-hand side of this formula is bounded above by $K|z|^{-1-\delta}$ for some $K > 0$ independent of ξ, because the two Dirac operators depend on ξ in the same way, hence their difference does not depend on it at all. Introducing the coordinate $w = z(\xi - \xi_l)$, this becomes $K|w|^{-1-\delta}|\xi - \xi_l|^{1+\delta}$. Therefore, it is sufficient to prove

$$\int_{\mathbf{C} \setminus \Delta(0, |\xi-\xi_l|/\varepsilon_0)} |w|^{-2-2\delta} |\xi - \xi_l|^{2+2\delta} \left| \hat{\sigma}_k^l(z, \xi) \right|^2_{|dz|^2, h} |\xi - \xi_l|^{-2} |dw|^2$$

$$\leq K |\xi - \xi_l|^{2+2\delta} \int_{\mathbf{C}} \left| \hat{\sigma}_k^l(z, \xi) \right|^2_{|dz|^2, h} |\xi - \xi_l|^{-2} |dw|^2,$$

for a suitable $K > 0$, or more simply

$$\int_{\mathbf{C} \setminus \Delta(0, |\xi-\xi_l|/\varepsilon_0)} |w|^{-2-2\delta} \left| \hat{\sigma}_k^l(z, \xi) \right|^2_{|dz|^2, h} |dw|^2$$

$$(4.68) \qquad \qquad \leq K \int_{\mathbf{C}} \left| \hat{\sigma}_k^l(z, \xi) \right|^2_{|dz|^2, h} |dw|^2.$$

This goes similarly to (4.58): because in the coordinate $w = h_\xi^{-1} z$ the spinor $|\xi - \xi_l|^{2-2\alpha_k^\infty} \hat{\sigma}_k^l(z, \xi)$ is independent of ξ (see Step 1) and h and h^∞ are mutually bounded, it boils down to

$$\int_{\mathbf{C} \setminus \Delta(0, |\xi-\xi_l|/\varepsilon_0)} |w|^{-2-2\delta} \left| (h_\xi^* \hat{\sigma}_k^l)(w) \right|^2_{|dz|^2, h^\infty} |dw|^2$$

$$\leq K \int_{\mathbf{C}} \left| (h_\xi^* \hat{\sigma}_k^l)(w) \right|^2_{|dz|^2, h^\infty} |dw|^2.$$

Now remark that $h_\xi^* \hat{\sigma}_k^l \in H^1(\mathbf{C}, |dw|^2, h^\infty)$ implies in particular that $h_\xi^* \hat{\sigma}_k^l \in L^2(\mathbf{C}, |dw|^2, h^\infty)$. Furthermore, near the origin $|w|^{-1-\delta} h_\xi^* \hat{\sigma}_k^l \in L^2_{loc}(|dw|^2, h^\infty)$ provided that $\delta < \alpha_k^\infty$. Hence $|w|^{-1-\delta} h_\xi^* \hat{\sigma}_k^l \in L^2(\mathbf{C}, |dw|^2, h^\infty)$, and

$$K = 2 \frac{\left\| |w|^{-1-\delta} h_\xi^* \hat{\sigma}_k^l \right\|^2_{L^2(\mathbf{C}, |dw|^2, h^\infty)}}{\left\| h_\xi^* \hat{\sigma}_k^l \right\|^2_{L^2(\mathbf{C}, |dw|^2, h^\infty)}}$$

has the desired property (4.68).   □

This has the following consequence.

LEMMA 4.37. *As $\xi \to \xi_l$, we have the estimate*

$$\left| \|\hat{\sigma}(z,\xi)\|_{L^2}^2 - \left\|\hat{\pi}_\xi^H \hat{\sigma}(z,\xi)\right\|_{L^2}^2 \right| \leq K|\xi - \xi_l|^{2\delta} \|\hat{\sigma}(z,\xi)\|_{L^2}^2$$

for some $K > 0$ independent of ξ.

PROOF. Again as in Lemma 4.34, it is sufficient to bound

$$\left\|\hat{\sigma}(z,\xi) - \hat{\pi}_\xi^H \hat{\sigma}(z,\xi)\right\|_{L^2}^2$$

as in the lemma, where

$$\hat{\pi}_\xi^H \hat{\sigma}(z,\xi) = (\mathrm{Id} - \dbar_\xi G_\xi \dbar_\xi^*)\hat{\sigma}(z,\xi)$$

is the $\dbar_\xi^*$-harmonic representative of $\hat{\sigma}(\xi)$. Thus by Lemma 2.27 we have for the norm of the difference

$$\left\|\dbar_\xi G_\xi \dbar_\xi^* \hat{\sigma}(z,\xi)\right\|_{L^2}^2 \leq K|\xi - \xi_l|^{-2} \left\|\dbar_\xi^* \hat{\sigma}(z,\xi)\right\|_{L^2}^2$$

and we conclude using Lemma 4.36.   □

We are now ready to finish the proof of Theorem 4.35: by Lemma 4.37, as $\xi \to \xi_l$ the norm of the harmonic representative of the spinor $\hat{\sigma}(z,\xi)$ verifies

$$\frac{\left\|\hat{\pi}_\xi^H \hat{\sigma}(\xi)\right\|_{L^2}^2}{\|\hat{\sigma}(\xi)\|_{L^2}^2} \longrightarrow 1.$$

On the other hand, since the support of χ in the coordinate w is $\mathbf{C} \setminus \Delta(0, |\xi - \xi_l|/\varepsilon_0)$, and these sets exhaust $\mathbf{C}$ as $\xi \to \xi_l$, we have that

$$\frac{\|\hat{\sigma}(\xi)\|_{L^2}^2}{\|\hat{\sigma}_k^l(\xi)\|_{L^2}^2} \longrightarrow 1.$$

By (4.66) the L^2-norm of $\hat{\sigma}_k^l(z,\xi)$ as measured by the harmonic metric h satisfies

$$c|\xi - \xi_l|^{-2+2\alpha_k^\infty} \leq \int_{\mathbf{C}} \left|\hat{\sigma}_k^l(z,\xi)\right|_{|dz|^2,h}^2 |dz|^2 \leq C|\xi - \xi_l|^{-2+2\alpha_k^\infty}.$$

Putting together all this, we obtain (4.67), so that on the component of $\hat{E}$ near ξ_l on which the transformed Higgs field has eigenvalue $-\lambda_k^\infty$, the parabolic weight of the induced extension is $-1 + \alpha_k^\infty$.   □

4.7. The topology of the transformed bundle

In this section, we compute the topology of the underlying holomorphic bundle $^i\hat{\mathcal{E}}$ of the transformed Higgs bundle (see (4.8)) relative to its extension over the punctures given in Section 4.4. We then deduce the topology of the transformed Higgs bundle relative to its transformed extension given by Definition 3.11. We recall that we have denoted
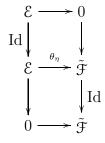
$$(4.69) \qquad \hat{r} = \sum_{p \in P} rk(Res(\theta, p))).$$

The result we wish to show is the following:

THEOREM 4.38. *The rank of $^i\hat{\mathcal{E}}$ is equal to $\hat{r}$, whereas its degree is equal to $\hat{r} + deg(\mathcal{E}) + r$, where r and $deg(\mathcal{E})$ are the rank and degree of $\mathcal{E}$, respectively.*

Notice that it gives in particular (1) of Theorem 1.32.

PROOF. Recall that we have denoted by $\mathcal{E}$ the sheaf of holomorphic sections of the bundle $\mathcal{E}$ underlying the original Higgs bundle; $\mathcal{F}$ was defined as a sheaf of meromorphic sections of $\mathcal{E} \otimes \Omega^{1,0}$ having singularities at $P \cup \{\infty\}$ with singular parts in prescribed spaces (see Subsection 4.3.1); and finally $\tilde{\mathcal{F}} = \pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{O}_{\widehat{\mathbf{CP}}^1}(1)$. By hypothesis, θ (and so θ_η for any η) is holomorphic with respect to the holomorphic structure $\bar{\partial}^\mathcal{E}$. Thus we may consider the holomorphic chain complex

$$
\begin{array}{ccc}
\mathcal{E} & \longrightarrow & 0 \\
{\scriptstyle\mathrm{Id}}\downarrow & & \downarrow \\
\mathcal{E} & \xrightarrow{\theta_\eta} & \tilde{\mathcal{F}} \\
\downarrow & & \downarrow{\scriptstyle\mathrm{Id}} \\
0 & \longrightarrow & \tilde{\mathcal{F}}
\end{array}
$$

in $\eta \in \widehat{\mathbf{CP}}^1$. The hypercohomology long exact sequence associated to it yields the exact sequence of cohomology spaces

$$0 \longrightarrow H^0(\mathbf{CP}^1, \mathcal{E}) \xrightarrow{\theta_\eta} H^0(\mathbf{CP}^1, \tilde{\mathcal{F}}) \longrightarrow \mathbb{H}^1(\mathcal{E} \xrightarrow{\theta_\eta} \tilde{\mathcal{F}})$$

$$(4.70) \qquad \longrightarrow H^1(\mathbf{CP}^1, \mathcal{E}) \xrightarrow{\theta_\eta} H^1(\mathbf{CP}^1, \mathcal{F}) \longrightarrow 0,$$

since we have seen that $\mathbb{H}^0(\mathcal{E} \xrightarrow{\theta_\eta} \tilde{\mathcal{F}}) = \mathbb{H}^2(\mathcal{E} \xrightarrow{\theta_\eta} \tilde{\mathcal{F}}) = 0$. All of the spaces in this exact sequence come with a natural holomorphic structure over $\widehat{\mathbf{CP}}^1$:

- the cohomology spaces of $\mathcal{E}$ because this latter is trivial over $\widehat{\mathbf{CP}}^1$
- those of $\tilde{\mathcal{F}}$ because this latter is the tensor product of a trivial vector bundle over $\widehat{\mathbf{CP}}^1$ and $\mathcal{O}_{\widehat{\mathbf{CP}}^1}(1)$
- finally, $\mathbb{H}^1(\mathcal{E} \xrightarrow{\theta_\bullet} \tilde{\mathcal{F}}) = \hat{V}_\bullet$ has its holomorphic structure $\bar{\partial}^{\hat{\mathcal{E}}}$ induced by $\hat{\mathrm{d}}^{0,1}$, extended to the singularities in Section 4.4 by the induced extension ${}^i\hat{\mathcal{E}}$.

Moreover, all of the maps in the exact sequence (4.70) vary holomorphically in $\eta \in \widehat{\mathbf{CP}}^1$ with respect to these structures and extensions: this follows from the definition of $\tilde{\mathcal{F}}$ and that of the induced extension. Therefore, it induces an exact sequence of the sheaves over $\widehat{\mathbf{CP}}^1$ of holomorphic sections of the corresponding cohomology spaces:

$$0 \longrightarrow \mathcal{O}(H^0(\mathcal{E})) \xrightarrow{\theta_\eta} \mathcal{O}(H^0(\tilde{\mathcal{F}})) \longrightarrow \mathcal{O}({}^i\hat{\mathcal{E}})$$
$$\longrightarrow \mathcal{O}(H^1(\mathcal{E})) \longrightarrow \mathcal{O}(H^1(\mathcal{F})) \longrightarrow 0,$$

where $\mathcal{O}$ stands to denote the sheaf of regular sections on $\widehat{\mathbf{CP}}^1$ with respect to the above mentioned holomorphic structures. By additivity of the Chern character, we deduce the equality

$$(4.71) \qquad ch({}^i\hat{\mathcal{E}}) = ch(\mathcal{O}(\widehat{\mathbf{CP}}^1, H^0(\tilde{\mathcal{F}}))) - ch(\mathcal{O}(\widehat{\mathbf{CP}}^1, H^1(\tilde{\mathcal{F}})))$$

$$(4.72) \qquad - ch(\mathcal{O}(\widehat{\mathbf{CP}}^1, H^0(\mathcal{E}))) + ch(\mathcal{O}(\widehat{\mathbf{CP}}^1, H^1(\mathcal{E})))$$

in $H^*(\widehat{\mathbf{CP}}^1)$. Put $\pi = \pi_2$, the projection onto the second factor in $\mathbf{CP}^1 \times \widehat{\mathbf{CP}}^1$. One has direct image sheaves $\pi_*\mathcal{E}$ and $\pi_*\tilde{\mathcal{F}}$ on $\widehat{\mathbf{CP}}^1$ defined by

$$\pi_*\mathcal{E}|_U = \mathcal{O}(U, H^0(\mathbf{CP}^1, \mathcal{E}))$$
$$\pi_*\tilde{\mathcal{F}}|_U = \mathcal{O}(U, H^0(\mathbf{CP}^1, \tilde{\mathcal{F}}))) = \mathcal{O}(U, H^0(\mathbf{CP}^1, \mathcal{F})) \otimes \mathcal{O}_{\widehat{\mathbf{CP}}^1}(1)(U),$$

for any open set $U \in \mathbf{CP}^1$, and one can form the "virtual" sheaves

$$\pi_!\mathcal{E}|_U = \mathcal{O}(U, H^0(\mathbf{CP}^1, \mathcal{E})) - \mathcal{O}(U, H^1(\mathbf{CP}^1, \mathcal{E}))$$
$$\pi_!\tilde{\mathcal{F}}|_U = \mathcal{O}(U, H^0(\mathbf{CP}^1, \tilde{\mathcal{F}})) - \mathcal{O}(U, H^1(\mathbf{CP}^1, \tilde{\mathcal{F}})).$$

Again by additivity of the Chern character, the right-hand-side of (4.71) is equal to $ch(\pi_!\tilde{\mathcal{F}})$, which is in turn equal to

$$\pi_*(ch(\tilde{\mathcal{F}}) \cup Td(T_\pi)),$$

by the Grothendieck-Riemann-Roch theorem, where

$$T_\pi = T(\mathbf{CP}^1 \times \widehat{\mathbf{CP}}^1) - \pi^* T\widehat{\mathbf{CP}}^1 = \pi_1^* T\mathbf{CP}^1$$

is the relative tangent bundle of π, and Td stands for its Todd class. Moreover, π_* is just evaluation on the fundamental cycle of $\mathbf{CP}^1$. Similarly, we see that (4.72) is just

$$-ch(\pi_! \mathcal{E}) = -\pi_*(ch(\mathcal{E}) \cup Td(T_\pi)),$$

and thus we obtain

(4.73) $$ch({}^i\hat{\mathcal{E}}) = [(ch(\tilde{\mathcal{F}}) - ch(\mathcal{E})) \cup Td(\pi_1^* T\mathbf{CP}^1)]/[\mathbf{CP}^1].$$

Now we have

$$ch(\mathcal{E}) = r + c_1(\mathcal{E})$$

$$ch(\tilde{\mathcal{F}}) = \left[r + c_1(\mathcal{E}) + h \sum_{p \in P} rk(Res(\theta, p)) \right] (1 + \hat{h})$$

$$Td(T\mathbf{CP}^1) = Td(\mathcal{O}_{\mathbf{CP}^1}(2)) = 1 + h,$$

where r is the rank of the bundle $\mathcal{E}$, $c_1(\mathcal{E})$ is its first Chern class, and h and $\hat{h}$ are the hyper-plane classes of $\mathbf{CP}^1$ and $\widehat{\mathbf{CP}}^1$ respectively. Putting all this together, we obtain

$$ch(\tilde{\mathcal{F}}) - ch(\mathcal{E}) = \hat{r}h + [r + c_1(\mathcal{E}) + \hat{r}]\hat{h},$$

and plugging this into (4.73),

(4.74) $$ch({}^i\hat{\mathcal{E}}) = \hat{r} + [r + deg(\mathcal{E}) + \hat{r}]\hat{h},$$

as we wished.   □

We are now ready to pass back to the transformed extension of the Higgs bundle introduced in Definition 3.11, hence establishing points (2), (5) and (8) of Theorem 1.32.

COROLLARY 4.39. *The parabolic weights of the transformed Higgs bundle endowed with its transformed extension are α_k^∞ at the logarithmic punctures (on the same subspace as in Theorem 4.35) and α_k^j at infinity (on the subspace in Theorem 4.32). The degree of the transformed Higgs bundle $\hat{\mathcal{E}}$ with respect to its transformed extension is equal to the degree of $\mathcal{E}$.*

PROOF. Recall from Theorems 4.35 and 4.32 that the parabolic weigths of the transformed Higgs bundle relative to the induced extensions considered in Subsections 4.4.1 and 4.4.2 are equal to $-1 + \alpha_k^\infty$ at the logarithmic punctures and to $-1 + \alpha_k^j$ at infinity. On the other hand, by Definition 3.11, the parabolic weights of the transformed Higgs bundle with respect to its transformed extension are required to have parabolic weights between 0 and 1. This means that a local holomorphic trivialisation of the singular component of the transformed extension $\hat{\mathcal{E}}$ near the puncture ξ_l is

$$(\xi - \xi_l)\hat{\sigma}_k^l(\xi),$$

where $\hat{\sigma}_k^l(\xi)$ is the local holomorphic section of the extension ${}^i\hat{\mathcal{E}}$ at ξ_l defined in Subsection 4.4.1 and $k \in \{1 + a_l, \ldots, a_{l+1}\}$. On the regular component of $\hat{\mathcal{E}}|_{\xi_l}$ the harmonic representatives have bounded norm, which gives 0 parabolic weight. Therefore on this component one does not need to change the trivialisation. Similarly, a local holomorphic frame of $\hat{\mathcal{E}}$ near infinity can be expressed by

$$\xi^{-1}\hat{\sigma}_k^\infty(\xi),$$

where $\hat{\sigma}_k^\infty$ is the local holomorphic section of the extension ${}^i\hat{\mathcal{E}}$ at infinity defined in Subsection 4.4.2 localized near p_j for some $j \in \{1, \ldots, n\}$, and $k \in \{r_j + 1, \ldots, r\}$. Clearly, this way we increased all non-vanishing parabolic weights by 1. On the other hand, by Remark 1.12 even if the algebraic geometric degree of the bundle depends on the choice of extensions, the parabolic degree with respect to a fixed metric is independent of them, because it is always 0. Recall from Definition 1.11 that

$$deg_{par}({}^i\hat{\mathcal{E}}) = deg({}^i\hat{\mathcal{E}}) + \sum_{j \in \{1,\ldots,n,\infty\}} \sum_{k=r_j+1}^{r} (-1 + \alpha_k^j).$$

This quantity is therefore equal to

$$(4.75) \qquad deg_{par}(\hat{\mathcal{E}}) = deg(\hat{\mathcal{E}}) + \sum_{j \in \{1,\ldots,n,\infty\}} \sum_{k=r_j+1}^{r} \alpha_k^j.$$

Putting these expressions together, we deduce that

$$deg(\hat{\mathcal{E}}) = deg({}^i\hat{\mathcal{E}}) - \hat{r} - r,$$

where we recall again that we have defined

$$\hat{r} = \sum_{j=1}^{n} rk(Res(\theta, p_j)).$$

Using formula (4.74) we get

(4.76) $$deg(\hat{\mathcal{E}}) = deg(\mathcal{E}).$$

$\square$

CHAPTER 5

The inverse transform

In this chapter we construct the inverse of the transform introduced in the previous chapters. In line with the properties of the ordinary Fourier transform and its algebraic counterparts, the inverse is defined by a formula which only differs from the transform in a sign.

Recall from Section 3.1 that the transformed flat connection on $\hat{E}_\bullet = L^2 H^1(D_\bullet)$ is defined by the L^2-orthogonal projection of $\hat{\mathrm{d}} - z\mathrm{d}\xi\wedge$. For any parabolic vector bundle with integrable connection (F, D^F, h^F) on $\hat{\mathbf{C}}$ satisfying the conditions of Section 1.2 (i.e. having a finite number of simple poles in finite points and a second-order pole at infinity, such that the eigenvalues and parabolic weights meet the conditions imposed in Theorem 1.17), one can define the inverse transformed bundle with integrable connection $(\check{F}, \check{D}^F, \check{h}^F)$ on $\mathbf{C}$ by a procedure similar to the one defining $(\hat{E}, \hat{D}, \hat{h})$ starting from (E, D, h): namely, consider the deformation

(5.1) $$D_z^F = D^F + z\mathrm{d}\xi\wedge$$

of the connection parametrized by z in $\mathbf{C}$ minus a finite set, and let $\check{F}_z$ be the first L^2-cohomology of

$$F \xrightarrow{D_z^F} \Omega_{\hat{C}}^1 \otimes F \xrightarrow{D_z^F} \Omega_{\hat{C}}^2 \otimes F.$$

These vector spaces are of the same dimension and form a smooth vector bundle over $\mathbf{C}$ minus a finite number of points. The critical points are easily seen to be the opposites of the eigenvalues of the second-order term of D^F at infinity. The proof goes similarly to the case of the direct transform. We also define the Hilbert bundle $\check{H}$ over $\mathbf{C}$, the L^2-metric $\check{h}$ and the orthogonal projection $\check{\pi}_z : \check{H}_z \to \check{F}_z$ in an analogous manner as in Section 3.1. Next, let the inverse transformed integrable connection $\check{D}^F$ be defined by the parallel sections $\check{\pi}_z(e^{(z_0-z)\xi}\phi_{z_0}(\xi))$ for any harmonic section $\phi_{z_0}(\xi) \in \check{F}_{z_0}$. Equivalently, denoting by $\check{\mathrm{d}}$ the trivial connection with respect to w in the trivial Hilbert bundle $\check{H}$, the

inverse transformed flat connection can be given by the formula

$$(5.2) \qquad \check{D}^F = \check{\pi}_z(\check{d} + z d\xi),$$

as it can be seen by the argument given in Section 3.1, changing signs. Finally, we define the inverse transformed metric $\check{h}^F$ on the fiber $\check{F}_{z_0}$ again as the L^2-norm on $\hat{\mathbf{C}}$ of a $D^F_{z_0}$-harmonic representative. We can now state the

THEOREM 5.1. *The inverse transform of* $\mathsf{N} : (E, D, h) \mapsto (\hat{E}, \hat{D}, \hat{h})$ *is* $\mathsf{N}^{-1} : (F, D^F, h^F) \mapsto (\check{F}, \check{D}^F, \check{h}^F)$. *In different terms, for any bundle with integrable connection and harmonic metric* (E, D, h) *satisfying the conditions of Section 1.2 and the ones imposed in Theorem 1.17, there exists a canonical Hermitian bundle isomorphism* ω *between* $\check{\hat{E}}$ *and* E *such that* $\omega^* D = \check{\hat{D}}$.

REMARK 5.2. *As one can check using the transform on the level of singularity parameters described in Theorem 1.17, the assumptions (1) and (2) of that theorem are symmetric, in the sense that if they are fulfilled by* (E, D) *than the same is true for* $(\hat{E}, \hat{D})$. *Therefore, the transform* ˘ *can be applied to this latter, so the affirmation of the theorem has a meaning.*

PROOF. The proof is done in four steps: first, we prove that the fibers over $0 \in \mathbf{C}$ of E and $\check{\hat{E}}$ are canonically isomorphic. Next we show the same thing for the other fibers. Then we prove that the integrable connections are the same, and finally we establish equality of the harmonic metrics and parabolic structures.

Step 1. Consider the product manifold $\mathbf{C} \times \hat{\mathbf{C}}$, and let π_1 and π_2 be the projection to the first and second factor, respectively. Denote by E the pull-back vector bundle $\pi_1^* E$ on the product, and define the connection $D = \pi_1^* D - \xi dz - z d\xi$. Notice that on the fiber $\mathbf{C} \times \{\xi_0\}$ this just gives the deformation D_{ξ_0}. Now form the double complex

$$\mathcal{D}^{p,q} = \Omega^p_{\mathbf{C}} \otimes \Omega^q_{\hat{\mathbf{C}}}(E),$$

where $\Omega^p_{\mathbf{C}}$ (respectively $\Omega^q_{\hat{\mathbf{C}}}$) denote smooth p-forms (smooth q-forms) on $\mathbf{C}$ ($\hat{\mathbf{C}}$); and with differentials $d_1 = D_\xi, d_2 = \hat{d} - z d\xi \wedge$. Remark that these differentials commute (in the graded sense), and their sum is just D. The desired isomorphism will result from the study of the spectral sequences corresponding to the two different filtrations of this double complex.

Namely, consider the first filtration of $\mathcal{D}$: the first page of the corresponding spectral sequence $E_1^{\bullet,\bullet}$ is

$$(5.3) \qquad\qquad 0 \qquad \Omega^2_{\hat{\mathbf{C}}} \otimes \hat{E} \qquad 0$$

$$\mathrm{d}_2^\sharp \uparrow$$

$$0 \qquad \Omega^1_{\hat{\mathbf{C}}} \otimes \hat{E} \qquad 0$$

$$\mathrm{d}_2^\sharp \uparrow$$

$$0 \qquad\quad \hat{E} \qquad\quad 0$$

where $\mathrm{d}_2^\sharp$ stands for the operator induced by d_2. More precisely, this operator is obtained as follows. Consider for example a local section of $\hat{E}$: if $B(\xi_0)$ is an open ball in $\hat{\mathbf{C}}$, it is given by cohomology classes $[\phi_\xi]$ in $L^2 H^1(D_\xi)$ changing smoothly with $\xi \in B(\xi_0)$. Here $\phi_\xi = \phi_\xi(z)$ is a global L^2-section of E over $\mathbf{C}$, in the kernel of $\hat{\phi}_\xi^*$. In particular, $D_\xi \phi_\xi = 0$, and since the two differentials commute, we then have $D_\xi \circ \mathrm{d}_2 \phi_\xi = 0$. In other words, $\mathrm{d}_2 \phi_\xi$ is a d_1-closed section of $\mathcal{D}^{1,1}$ on $\mathbf{C} \times B(\xi_0)$; hence we may consider its cohomology class with respect to d_1, and letting ξ vary these give a section of $\Omega^1 \otimes \hat{E}$ over $B(\xi_0)$, which is by definition $\mathrm{d}_2^\sharp[\phi_\xi]$. Now remark that under the isomorphism of the first L^2-cohomology of the elliptic complex (2.24) and the space of $\hat{\phi}_\xi$-harmonic sections given in Theorem 2.21, this induced connection goes over to $\hat{D}$ defined in Section 3.1; in other words, under these identifications $\mathrm{d}_2^\sharp = \hat{D}$. Moreover, the connection $\hat{D}$ also satisfies the conditions of Section 1.2. Therefore, by Chapter 2 and Section 2.3 the L^2-cohomology of $\hat{D} = \hat{D}_0$ is non-trivial only in degree 1, and so when passing to the second page $E_2^{\bullet,\bullet}$ of the spectral sequence, we obtain by definition $E_2^{1,1} = \check{\hat{E}}_0$ and all other terms equal to 0. In particular, the spectral sequence collapses at the second page, and the total cohomology of the double complex is canonically isomorphic to $\check{\hat{E}}_0$ in degree 2 and vanishes in all other degrees.

Consider now the second filtration of $\mathcal{D}$: in order to form the first page $\tilde{E}_1^{\bullet,\bullet}$ of the corresponding spectral sequence, we first take cohomology on each column of the double complex with respect to $\mathrm{d}_2 = \hat{\mathrm{d}} - z\mathrm{d}\xi$,

and so it is equal to

$$(5.4) \quad \begin{array}{ccc} 0 & 0 & 0 \\[2em] 0 & 0 & 0 \\[2em] L^2(\mathbf{C}, E)e^{z\xi} \xrightarrow{\mathrm{d}_1^{\sharp}} L^2(\mathbf{C}, \Omega^1_{\mathbf{C}} \otimes E)e^{z\xi} \xrightarrow{\mathrm{d}_1^{\sharp}} L^2(\mathbf{C}, \Omega^2_{\mathbf{C}} \otimes E)e^{z\xi}. \end{array}$$

In words: for example, the $(0,0)$-term consists of L^2-sections of E on $\mathbf{C} \times \hat{\mathbf{C}}$ which are a product of an arbitrary section of E on $\mathbf{C}$ and the function $e^{z\xi}$. Now notice that the only possibility for a non-zero section of this form to be in L^2 on $\{z\} \times \hat{\mathbf{C}}$ is for $z = 0$. Put another way, the cohomology along the slices $\{z\} \times \hat{\mathbf{C}}$ vanishes for all $z \neq 0$. Hence we may replace the double complex $\mathcal{D}$ without changing the spectral sequence associated with this filtration (and so the total cohomology), by the double complex (germ $\mathcal{D}$) whose component of bidegree (p, q) is the space of L^2-forms with values in E of bidegree (p, q) defined on $V_0 \times \hat{\mathbf{C}}$ for any neighborhood V_0 of $0 \in \mathbf{C}$, and where we identify such forms if they coincide on an arbitrary neighborhood of $\{0\} \times \hat{\mathbf{C}}$. Of course, the differentials of this new double complex are induced by those of $\mathcal{D}$ in a trivial way.

The idea now is to consider the spectral sequence (germ E) corresponding to the first filtration of (germ $\mathcal{D}$): by the general theory of spectral sequences, this will then abut to the total cohomology of (germ $\mathcal{D}$), which is, as we saw in the previous paragraph, equal to that of $\mathcal{D}$, that is to $\check{E}_0$. First trivialize E in V_0: this just means that we identify the total space of the bundle with $V_0 \times E_0$. Since the vector bundle E on $\mathbf{C} \times \hat{\mathbf{C}}$ is just the pull-back of E on $\mathbf{C}$, this also gives an identification of $E \to V_0 \times \hat{\mathbf{C}}$ with the trivial bundle $(V_0 \times \hat{\mathbf{C}}) \times E_0$. Without loss of generality we may assume $0 \notin P$, so for V_0 sufficiently small the connection D can also be taken by a gauge transformation $\tilde{g}$ to the trivial one. Thus in this trivialisation and gauge we have $\mathrm{d}_1 = \mathrm{d} - \xi \mathrm{d}z$ where d stands for the trivial connection in the z direction. The first page (germ E)$_1^{\bullet, \bullet}$ is then equal to the cohomology spaces

with respect to this differential:

$$(5.5) \qquad \Omega^2_{\hat{\mathbf{C}}} \otimes L^2(\hat{\mathbf{C}}, E_0)e^{z\xi} \qquad 0 \qquad 0$$

$$\mathrm{d}^\sharp_2 \Big\uparrow$$

$$\Omega^1_{\hat{\mathbf{C}}} \otimes L^2(\hat{\mathbf{C}}, E_0)e^{z\xi} \qquad 0 \qquad 0$$

$$\mathrm{d}^\sharp_2 \Big\uparrow$$

$$L^2(\hat{\mathbf{C}}, E_0)e^{z\xi} \qquad 0 \qquad 0,$$

where, as before, $L^2(\hat{\mathbf{C}}, E_0)e^{z\xi}$ stands to denote functions with values in E_0 of the form $\gamma(\xi)e^{z\xi}$ but this time on $V_0 \times \hat{\mathbf{C}}$, and the L^2 condition now only implies that γ must be rapidly decreasing as $|\xi| \to \infty$. The next remark is that since we only have terms in degree $p = 0$, the differential induced by d_2 is just itself: indeed, it is by definition d_2 modulo the image of d_1, but this latter vanishes for $p = 0$. Thus, in order to obtain the second page (germ E)$^{\bullet,\bullet}_2$ of the spectral sequence, we take cohomology with respect to $\mathrm{d}_2 = \hat{\mathrm{d}} - z\mathrm{d}\xi \wedge$. Notice that via the gauge transformation $e^{-z\xi}$ the whole picture can be rephrased as the de Rham cohomology of rapidly decreasing sections σ on $\hat{\mathbf{C}}$ with values in E_0, which is similar to compactly supported de Rham cohomology. Therefore in (germ E)$^{\bullet,\bullet}_2$ all elements except for the one corresponding to bidegree $(0, 2)$ vanish, and this latter is canonically isomorphic to E_0 via mapping an element $\gamma_0 \in E_0$ into the germ

$$[\gamma_0\chi(\xi)e^{z\xi}\mathrm{d}\xi \wedge \mathrm{d}\bar{\xi}],$$

where χ is a fixed exponentially decreasing bump-function on $\hat{\mathbf{C}}$ with integral (with respect to the volume form $|\mathrm{d}\xi|^2$) equal to 1, and $[.]$ stands to denote the de Rham cohomology class of exponentially decreasing forms on $\hat{\mathbf{C}}$ with values in E_0. Conversely, for an arbitrary class $[\gamma_0(\xi)e^{z\xi}\mathrm{d}\xi \wedge \mathrm{d}\bar{\xi}]$ where $\gamma_0(\xi)e^{z\xi}$ is a germ of exponentially decreasing functions on $\hat{\mathbf{C}}$ with values in E_0 and in the kernel of $\mathrm{d}_1 = (\mathrm{d} - \xi\mathrm{d}z)$, we may define

$$[\gamma_0(\xi)e^{z\xi}\mathrm{d}\xi \wedge \mathrm{d}\bar{\xi}] \mapsto eval_{z=0} \int_{\hat{\mathbf{C}}} \gamma_0(\xi)e^{z\xi}|\mathrm{d}\xi|^2$$

$$(5.6) \qquad\qquad\qquad = \int_{\hat{\mathbf{C}}} \gamma_0(\xi)|\mathrm{d}\xi|^2 \in E_0$$

and verify readily that it is independent of the section representing a cohomology class. The fact that E_0 and $\check{\hat{E}}_0$ are canonically isomorphic

now follows from the fact that they are both canonically isomorphic to (different gradings of) the total cohomology of the double complex $\mathcal{D}$.

Step 2. The first thing to do is to describe explicitly the isomorphism obtained above. Let $\left[\check{\hat{\delta}}_0\right]$ be an element in $\check{\hat{E}}_0$: it is a class in the cohomology space $\mathrm{E}_2^{1,1}$ in the spectral sequence corresponding to the first filtration of $\mathcal{D}$. Hence it is represented by a $(1,1)$-form $\check{\hat{\delta}}_0(z;\xi)$ over $\mathbf{C} \times \hat{\mathbf{C}}$ such that

    (1) $(D - \xi \mathrm{d}z\wedge)\check{\hat{\delta}}_0(z;\xi) = 0$
    (2) $(\hat{\mathrm{d}} - z\mathrm{d}\xi\wedge)^\sharp\check{\hat{\delta}}_0(z;\xi) = 0$; in other words, there exists a $(0,2)$-
        form $\gamma_0(z;\xi)$ over $\mathbf{C} \times \hat{\mathbf{C}}$ satisfying

$$D_\xi \gamma_0(z;\xi) = (\hat{\mathrm{d}} - z\mathrm{d}\xi\wedge)\check{\hat{\delta}}_0.$$

Concatenating the map

$$\left[\check{\hat{\delta}}_0\right] \mapsto \gamma_0(z;\xi)$$

with an analog of (5.6), namely

(5.7) $$[\gamma_0(z;\xi)] \mapsto eval_{z=0} \int_{\hat{\mathbf{C}}} \gamma_0(z;\xi)$$

we get the canonical isomorphism

$$\omega_0 : \left[\check{\hat{\delta}}_0\right] \mapsto \delta_0 = eval_{z=0} \int_{\hat{\mathbf{C}}} \gamma_0(z;\xi)$$

between $\check{\hat{E}}_0$ and E_0 provided by the previous step.

Fix now an arbitrary $z_0 \in \mathbf{C}$, and consider the double complex $\mathcal{D}_{z_0}$ having the same (p,q)-components as $\mathcal{D}$, but with differentials $\mathrm{d}_1 = D_\xi, \mathrm{d}_2 = \hat{\mathrm{d}} - (z - z_0)\mathrm{d}\xi\wedge$. In order to obtain the components of the first page $(\mathrm{E}_{z_0})_1^{\bullet,\bullet}$ of the spectral sequence corresponding to the first filtration of $\mathcal{D}_{z_0}$, we need to take cohomology with respect to d_1, hence these will be the same as those of $\mathcal{D}$ in (5.3), and the differentials will be induced by d_2. Now since z_0 is a constant, observe that for any local section $\phi_\xi(z) \in Ker\hat{\phi}_\xi^*$ in ξ of harmonic sections over $\mathbf{C}$ the relation

$$\mathrm{d}_2^\sharp\phi_\xi = [(\hat{\mathrm{d}} - (z - z_0)\mathrm{d}\xi\wedge)\phi_\xi] = [(\hat{\mathrm{d}} - z\mathrm{d}\xi\wedge)\phi_\xi] + z_0\mathrm{d}\xi \wedge \phi_\xi = \hat{D}_{z_0}(\phi_\xi),$$

holds, where $\hat{D}_{z_0}$ is the deformation of $\hat{D}$ introduced in (5.1). To get the second page of the spectral sequence, we take cohomology with respect to $\mathrm{d}_2^\sharp = \hat{D}_{z_0}$, and therefore if z_0 does not belong to the set of opposites of eigenvalues of the leading term of $\hat{D}$ then this is a finite-dimensional space, equal by definition to $\check{\hat{E}}_{z_0}$. Notice that by the results

of Subsection 4.5, the set of z_0 where this does not hold is exactly P, the set of singularities (at finite points) of E. Similarly, the second filtration of $\mathcal{D}_{z_0}$ gives rise to a spectral sequence whose first page is (analogously to (5.4))

$$0 \qquad\qquad 0 \qquad\qquad 0$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

$$L^2(\mathbf{C}, E)e^{(z-z_0)\xi} \xrightarrow{\mathrm{d}_1^\sharp} L^2(\mathbf{C}, \Omega_{\mathbf{C}}^1 \otimes E)e^{(z-z_0)\xi} \xrightarrow{\mathrm{d}_1^\sharp} L^2(\mathbf{C}, \Omega_{\mathbf{C}}^2 \otimes E)e^{(z-z_0)\xi}.$$

Hence the only fiber $\{z\} \times \hat{\mathbf{C}}$ over which these spaces are non-trivial is for $z = z_0$, so we may consider the double complex (germ $\mathcal{D}_{z_0}$) whose components are germs of forms in a neighborhood $V_{z_0} \times \hat{\mathbf{C}}$ of the fiber $\{z_0\} \times \hat{\mathbf{C}}$, two such germs being identified if they coincide in any such neighborhood, and with differentials coming from those of $\mathcal{D}_{z_0}$. As before, the spectral sequences corresponding to the second filtration of these double complexes agree starting from the first page, so in particular their total cohomologies are the same. Now, we pass back again to the first filtration and compute the spectral sequence of (germ $\mathcal{D}_{z_0}$) with respect to it: in a convenient trivialisation of E in V_0 and gauge, the first page is equal to

$$(5.8) \qquad \begin{array}{ccc} \Omega_{\hat{\mathbf{C}}}^2 \otimes L^2(\hat{\mathbf{C}}, E_{z_0})e^{z\xi} & 0 & 0 \\[2mm] \mathrm{d}_2^\sharp \uparrow & & \\[2mm] \Omega_{\hat{\mathbf{C}}}^1 \otimes L^2(\hat{\mathbf{C}}, E_{z_0})e^{z\xi} & 0 & 0 \\[2mm] \mathrm{d}_2^\sharp \uparrow & & \\[2mm] L^2(\hat{\mathbf{C}}, E_{z_0})e^{z\xi} & 0 & 0, \end{array}$$

with differentials given by $\mathrm{d}_2 = \hat{\mathrm{d}} - (z - z_0)\mathrm{d}\xi\wedge$. As in step 1, the second page therefore contains only one non-vanishing component: the one corresponding to bidegree $(0, 2)$, and it is canonically isomorphic to the vector space E_{z_0}; this proves that the vector spaces E_{z_0} and $\check{\hat{E}}_{z_0}$ are canonically isomorphic to each other. Again, an element $\left[\check{\hat{\delta}}_{z_0}\right]$ of $\check{\hat{E}}_{z_0}$ is represented by a $(1, 1)$-form $\check{\hat{\delta}}_{z_0}(z; \xi)$ over $\mathbf{C} \times \hat{\mathbf{C}}$ satisfying $(\hat{\mathrm{d}} - (z - z_0)\mathrm{d}\xi)^\sharp \check{\hat{\delta}}_{z_0}(z; \xi) = 0$, i.e. there exists a $(0, 2)$-form $\gamma_{z_0}(z; \xi)$

over $\mathbf{C} \times \hat{\mathbf{C}}$ with

$$D_\xi(\gamma_{z_0}(z;\xi)) = (\hat{\mathrm{d}} - (z - z_0)\mathrm{d}\xi\wedge)\check{\hat{\delta}}_{z_0}(z;\xi),$$

and an explicit way of describing the obtained isomorphism is given by

$$(5.9) \qquad \omega_{z_0} : \left[\check{\hat{\delta}}_{z_0}\right] \mapsto \delta_{z_0} = eval_{z=z_0} \int_{\hat{\mathbf{C}}} \gamma_{z_0}(z;\xi)$$

Step 3. By the previous points, we have that the bundle $\check{\hat{E}}$ is isomorphic to E via the isomorphisms $\omega_\bullet$. Now we prove that the integrable connection $\check{\hat{D}}$ on $\check{\hat{E}}$ is carried into D on E by this bundle isomorphism: for this, it is clearly sufficient to prove that any local parallel section for $\check{\hat{D}}$ is carried into a parallel section for D. For simplicity, we shall consider a local section near $w = 0$, but we will see that the proof does not use this.

For this purpose, we need to work on the product $\mathbf{C} \times \hat{\mathbf{C}} \times \mathbf{C}$, parametrized by (z, ξ, w); we keep on writing the variable w in lower index. We shall consider E as being a bundle over this space by pullback, without writing it out explicitly. Let $\left[\check{\hat{\delta}}_w\right]$ be a $\check{\hat{D}}$-parallel local section of $\check{\hat{E}}$. As in Step 2, such a section is represented by giving a global $(1,1)$-form $\check{\hat{\delta}}_w(z;\xi)$ of E on $\mathbf{C} \times \hat{\mathbf{C}}$ for each w in a neighborhood V_0 of $0 \in \mathbf{C}$, verifying

(1) $D_{\xi_0}\check{\hat{\delta}}_w(z;\xi) = 0$ for all fixed $w_0 \in V_0$ and $\xi_0 \in \hat{\mathbf{C}}$
(2) $(\mathrm{d}_2 - (z - w_0)\mathrm{d}\xi\wedge)^\sharp\check{\hat{\delta}}_w(z;\xi) = 0$ for all fixed $w_0 \in V_0$
(3) the section in w of the cohomology classes of the above elements is $\check{\hat{D}}$-parallel.

By Hodge theory, we may suppose that $\check{\hat{\delta}}_{w_0}(z;\xi_0)$ is the D_{ξ_0}-harmonic representative of $\left[\check{\hat{\delta}}_{w_0}|_{\mathbf{C}\times\{\xi_0\}}\right]$ and also that $\check{\hat{\delta}}_{w_0}(z;\xi)$ is the $\hat{D}_{w_0}$-harmonic representative of $\left[\check{\hat{\delta}}_{w_0}\right]$. This way we rephrase the above conditions as

(1) for all fixed $w_0 \in V_0$ and $\xi_0 \in \hat{\mathbf{C}}$ its restriction to the fiber $\mathbf{C} \times \{\xi_0\} \times \{w_0\}$ is in $\hat{E}_{\xi_0}$, that is $\hat{\phi}^*_{\xi_0}\check{\hat{\delta}}_{w_0}(z;\xi_0) = 0$
(2) for all fixed $w_0 \in V_0$ the global section in ξ of the above elements of $\hat{E}_\xi$ is in $\check{\hat{E}}_{w_0}$, in different terms $\check{\hat{\phi}}^*_{w_0}\check{\hat{\delta}}_{w_0}(z;\xi) = 0$
(3) and for all $w_0 \in V_0$, $\check{\pi}_w \circ (\check{\mathrm{d}} + \xi\mathrm{d}w\wedge)\check{\hat{\delta}}_w(z;\xi)|_{w=w_0} = 0$.

As before, (2) means that for all $w \in V_0$ there exists $\gamma_w(z;\xi) \in \Gamma(\mathbf{C} \times \hat{\mathbf{C}}, \Omega^{2,0} \otimes E)$ such that

(5.10) $$D_\xi \gamma_w(z;\xi) = (\hat{\mathrm{d}} - (z-w)\mathrm{d}\xi \wedge)\check{\bar{\delta}}_w(z;\xi);$$

and by Hodge theory, such a section can be defined by the formula

(5.11) $$\gamma_w(z;\xi) = G_\xi D_\xi^*(\hat{\mathrm{d}} - (z-w)\mathrm{d}\xi \wedge)\check{\bar{\delta}}_w(z;\xi),$$

where G_ξ is the Green's operator of $\check{\partial}_\xi^* \check{\partial}_\xi$. (Here we used that G_ξ is diagonal with respect to the decomposition $\Omega_{\mathbf{C}}^0 \oplus \Omega_{\mathbf{C}}^2$, a standard consequence of the fact that $\check{\partial}_\xi^* \check{\partial}_\xi$ is diagonal with respect to the same decomposition, which comes immediately from harmonicity of the metric.) Now by (5.9) and (5.10) we have

$$
\begin{aligned}
D\delta(w)|_{w=w_0} &= D\left(eval_{z=w} \int_{\hat{\mathbf{C}}} \gamma_w(z;\xi)\right)_{|w=w_0} \\
&= \int_{\hat{\mathbf{C}}} D\gamma_{w_0}(z;\xi)|_{z=w_0} + \check{\mathrm{d}}\gamma_w(w_0;\xi)|_{w=w_0} \\
&= \int_{\hat{\mathbf{C}}} \xi \mathrm{d}z \wedge \gamma_{w_0}(w_0;\xi) \\
&\quad + (\hat{\mathrm{d}} - (w_0 - w_0)\mathrm{d}\xi \wedge)\check{\bar{\delta}}_{w_0}(w_0;\xi) + \check{\mathrm{d}}\gamma_w(w_0;\xi)|_{w=w_0}
\end{aligned}
$$

(remember that $\check{\mathrm{d}}$ stands for the trivial connection with respect to w in the trivial Hilbert bundle $\check{H}$, whereas $\hat{\mathrm{d}}$ is the trivial connection with respect to ξ in the trivial Hilbert bundle $\hat{H}$). The integral of the middle term in this last formula vanishes by Stokes's theorem. Furthermore, on the diagonal $z = w$ of $\mathbf{C} \times \mathbf{C}$ we have $\mathrm{d}z = \mathrm{d}w$, so we are left with

$$\int_{\hat{\mathbf{C}}} (\check{\mathrm{d}} + \xi \mathrm{d}w \wedge)\gamma_{w_0}(w_0;\xi).$$

Applying to this quantity (5.11) and the commutation relations

(5.12) $$[\check{\mathrm{d}} + \xi \mathrm{d}w \wedge, \hat{\mathrm{d}} - (z-w)\mathrm{d}\xi \wedge] = 0 \qquad [\check{\mathrm{d}} + \xi \mathrm{d}w \wedge, D_\xi] = 0$$

we obtain

(5.13) $$\int_{\hat{\mathbf{C}}} G_\xi D_\xi^*(\hat{\mathrm{d}} - (z-w)\mathrm{d}\xi \wedge)(\check{\mathrm{d}} + \xi \mathrm{d}w \wedge)\check{\bar{\delta}}_{w_0}(w_0;\xi).$$

Consider now condition (3) above: denoting by $\tilde{\partial}_w$ and $\tilde{\partial}_w^*$ the positive and negative Dirac operators of the deformation $\hat{D} + w\mathrm{d}\xi$, moreover by $\hat{G}_w$ the Green's operator of $\tilde{\partial}_w^* \tilde{\partial}_w$, it can be rewritten as

$$(\mathrm{Id} - \tilde{\partial}_w \hat{G}_w \tilde{\partial}_w^*)(\check{\mathrm{d}} + \xi \mathrm{d}w \wedge)\check{\bar{\delta}}_w(z;\xi) = 0.$$

In order to finish the proof, it is sufficient to prove the commutation relation

(5.14) $$[\check{\mathrm{d}} + \xi\mathrm{d}w\wedge, \hat{\check{\phi}}_w] = 0.$$

Indeed, this then implies

$$[\check{\mathrm{d}} + \xi\mathrm{d}w\wedge, \hat{\check{\phi}}_w^*] = 0 \qquad\qquad [\check{\mathrm{d}} + \xi\mathrm{d}w\wedge, \hat{G}_w] = 0,$$

and interchanging $\check{\mathrm{d}} + \xi\mathrm{d}w\wedge$ turn by turn with $\hat{\check{\phi}}_{w_0}^*$, $\hat{G}_{w_0}$ and $\hat{\check{\phi}}_{w_0}$ using each time condition (2), we get

$$\begin{aligned}
(\check{\mathrm{d}} + \xi\mathrm{d}w\wedge)\check{\hat{\delta}}_{w_0}(w_0; \xi) &= (\check{\mathrm{d}} + \xi\mathrm{d}w\wedge)(\mathrm{Id} - \hat{\check{\phi}}_{w_0}\hat{G}_{w_0}\hat{\check{\phi}}_{w_0}^*)\check{\hat{\delta}}_{w_0}(w_0; \xi) \\
&= (\mathrm{Id} - \hat{\check{\phi}}_{w_0}\hat{G}_{w_0}\hat{\check{\phi}}_{w_0}^*)(\check{\mathrm{d}} + \xi\mathrm{d}w\wedge)\check{\hat{\delta}}_{w_0}(w_0; \xi) \\
&= 0,
\end{aligned}$$

and so (5.13) is equal to 0; but on the other hand it is just the expression for $D\delta(w)|_{w=w_0}$, and this shows that $\delta(w)$ is parallel in w_0. There remains to show (5.14): recall that $\hat{\check{\phi}}_w = \hat{D}_w - \hat{D}_w^*$, with

$$\hat{D}_w = \hat{\pi}_\xi(\check{\mathrm{d}} - (z - w)\mathrm{d}\xi).$$

Now the first relation in (5.12) and $\hat{\pi}_\xi = (\mathrm{Id} - \hat{\phi}_\xi G_\xi \hat{\phi}_\xi^*)$ combined with the second relation in (5.12) show that

$$[\check{\mathrm{d}} + \xi\mathrm{d}w\wedge, \hat{D}_w] = 0,$$

and we conclude.

Step 4. Here we wish to show that the double transformed metric $\check{\hat{h}}$ is equal to h. In Step 3 we have already shown that the flat connections D and $\check{\hat{D}}$ agree. On the other hand, using the results of Section 4.2 twice, we see that $\check{\hat{h}}$ is a harmonic metric for $\check{\hat{D}} = D$. Therefore by uniqueness (up to a constant) of the harmonic metric corresponding to an integrable connection, we get that $\check{\hat{h}} = h$.

An equivalent way of deducing the same assertion would be as follows: using again the already proved equality $\check{\hat{D}} = D$ and uniqueness of the harmonic metric, we will be done if we can prove that the unitary part $\check{\hat{D}}^+$ (with respect to $\check{\hat{h}}$) of the double transformed flat connection $\check{\hat{D}}$ is equal to D^+, the unitary part of D with respect to h. This can be done in a completely analogous way to Steps 1-3. The changes we have to make are the following: consider the double complex $\mathcal{D}_{z_0}^H$ having the same components as $\mathcal{D}_{z_0}$, but with differentials $\mathrm{d}_1 = D_\xi^H$ and

$\mathrm{d}_2 = \hat{\mathrm{d}} - z/2\mathrm{d}\xi \wedge -\bar{z}/2\mathrm{d}\bar{\xi}\wedge$. One establishes that these operators commute, therefore $\mathcal{D}^H_{z_0}$ really forms a double complex. We then see from (4.14) that the deformation

$$\hat{D}^H_w = \hat{D}^H + \frac{1}{2}w\mathrm{d}\xi \wedge +\frac{1}{2}\bar{w}\mathrm{d}\bar{\xi}\wedge$$

induced from the differential

$$\hat{\mathrm{d}} - \frac{1}{2}(z - w)\mathrm{d}\xi \wedge -\frac{1}{2}(\bar{z} - \bar{w})\mathrm{d}\bar{\xi}\wedge$$

is the natural deformation of the Higgs-bundle structure induced by the deformation $\hat{D}_w$. In concrete terms, they are related by the gauge transformation g^{-1}. Therefore the double transformed bundle $\check{\hat{E}}^H$ is isomorphic to $g^{-1}gE = E$, and the unitary connection

$$\check{D}^+ = \check{\pi}_w \circ \left(\check{\mathrm{d}} + \frac{\xi}{2}\mathrm{d}w \wedge +\frac{\bar{\xi}}{2}\mathrm{d}\bar{w}\wedge \right)$$

is identified to D^+ just as $\check{\hat{D}}$ with D, using the commutation relations

$$\left[\check{\mathrm{d}} + \frac{\xi}{2}\mathrm{d}w \wedge +\frac{\bar{\xi}}{2}\mathrm{d}\bar{w}\wedge, \hat{\mathrm{d}} - \frac{1}{2}(z - w)\mathrm{d}\xi \wedge -\frac{1}{2}(\bar{z} - \bar{z})\mathrm{d}\bar{\xi}\wedge \right] = 0,$$

$$\left[\check{\mathrm{d}} + \frac{\xi}{2}\mathrm{d}w \wedge +\frac{\bar{\xi}}{2}\mathrm{d}\bar{w}\wedge, D^H_\xi \right] = 0$$

instead of (5.12), which together imply the analog

$$\left[\check{\mathrm{d}} + \frac{\xi}{2}\mathrm{d}w \wedge +\frac{\bar{\xi}}{2}\mathrm{d}\bar{w}\wedge, \check{\hat{\partial}}^H_w \right] = 0$$

of (5.14) for the deformed Dirac operator

$$\check{\hat{\partial}}^H_w = D^H_w - (D^H_w)^*.$$

This then allows us to conclude equality of the unitary connections.

Since the Hermitian bundles $(\check{\hat{E}}, \check{\hat{h}})$ and (E, h) coincide, so do the flags of their parabolic structures in the singular points; as well as the parabolic weights, because they are supposed to be between 0 and 1, and there is a unique way of choosing holomorphic sections with such behaviors. $\qquad\square$

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